

# Associative algebraic approach to logarithmic CFT in the bulk: the continuum limit of the $\mathfrak{gl}(1|1)$ periodic spin chain, Howe duality and the interchiral algebra.

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## Abstract

We develop in this paper the principles of an associative algebraic approach to *bulk* logarithmic conformal field theories (LCFTs). We concentrate on the closed  $\mathfrak{gl}(1|1)$  spin-chain and its continuum limit – the  $c = -2$  symplectic fermions theory – and rely on two technical companion papers, *Continuum limit and symmetries of the periodic  $\mathfrak{gl}(1|1)$  spin chain* [arXiv:1112.3403] and *Bimodule structure in the periodic  $\mathfrak{gl}(1|1)$  spin chain* [arXiv:1112.3407].

Our main result is that the algebra of local Hamiltonians, the Jones–Temperley–Lieb algebra  $JTL_N$ , goes over in the continuum limit to a bigger algebra than  $\mathfrak{V}$ , the product of the left and right Virasoro algebras. This algebra,  $\mathcal{S}$  – which we call *interchiral*, mixes the left and right moving sectors, and is generated, in the symplectic fermions case, by the additional field  $S(z, \bar{z}) \equiv S_{\alpha\beta} \psi^\alpha(z) \bar{\psi}^\beta(\bar{z})$ , with a symmetric form  $S_{\alpha\beta}$  and conformal weights  $(1, 1)$ . We discuss in details how the Hilbert space of the LCFT decomposes onto representations of this algebra, and how this decomposition is related with properties of the finite spin-chain. We show that there is a complete correspondence between algebraic properties of finite periodic spin chains and the continuum limit.

An important technical aspect of our analysis involves the fundamental new observation that the action of  $JTL_N$  in the  $\mathfrak{gl}(1|1)$  spin chain is in fact isomorphic to an enveloping algebra of a certain Lie algebra, itself a non semi-simple version of  $\mathfrak{sp}_{N-2}$ . The semi-simple part of  $JTL_N$  is represented by  $U\mathfrak{sp}_{N-2}$ , providing a beautiful example of a classical Howe duality, for which we have a non semi-simple version in the full  $JTL_N$  image represented in the spin-chain. On the continuum side, simple modules over  $\mathcal{S}$  are identified with “fundamental” representations of  $\mathfrak{sp}_\infty$ .

## 1 Introduction

Our understanding of logarithmic conformal field theory (LCFT) has greatly improved recently thanks to a renewed focus on purely algebraic features and in particular a systematic study of Virasoro indecomposable modules.

Most of the developments have concerned boundary LCFTs, where two lines of attack have been pursued. The first one is rather abstract, and follows the pioneering work of Rohsiepe [1] and Gaberdiel [2], where, in particular, fusion is interpreted as the tensor product of the symmetry algebra [3, 4, 5]. The other [6, 7] is somehow more concrete, in that it relies on lattice regularizations, and exploits the (not entirely understood) similarity [8] between the properties of lattice models and

their conformally invariant continuum limits. The models involved in this second approach provide [9] representations of associative algebras such as the Temperley–Lieb algebra. It turns out that the representation theory of these algebras is, in a certain (categorical, see Sec. 2.4) sense [10], similar to the one of chiral algebras in LCFT. The structure of indecomposable modules and fusion rules can then be predicted from the analysis of the scaling limit of the lattice models [6, 7, 11]. The results are in perfect agreement (so far) with the conclusions of the first approach based on (considerably more involved) calculations in the Virasoro algebra [4, 12].

To be now a little more precise, the general philosophy of the lattice approach in the boundary case relies on the analysis of the microscopic model as a bi-module over two algebras. In physical terms, one of these algebras is generated by the local hamiltonian density, and the other is the ‘symmetry’ commuting with the hamiltonian densities. More precisely, the types of models we are interested in carry alternating fundamental representations of a super Lie algebra, and admit a single invariant nearest neighbor coupling, the ‘super equivalent’ of the Heisenberg interaction  $H_i \approx \vec{S}_i \cdot \vec{S}_{i+1}$ . The different  $H_i$ ’s turn out to obey some non-trivial algebraic relations, so that the models provide a representation of a certain abstract algebra  $A$  – the ordinary and boundary Temperley–Lieb algebras for instance.

Meanwhile, the local Hamiltonians are invariant under the super Lie algebra. In fact, they typically are invariant under a bigger algebra which is, in technical terms, the centralizer of  $A$ . We recall here that the centralizer  $\mathfrak{Z}_A$  of an algebra  $A$  on their common representation space  $V$  is the algebra of all intertwining operators  $\text{End}_A(V)$ , *i.e.*, the algebra of the maximum dimension such that  $[\mathfrak{Z}_A, A] = 0$ . For the open case of  $\mathfrak{gl}(m|n)$ -symmetric spin chains, this centralizer was dubbed  $\mathcal{A}_{m|n}$  in [10], where it was shown to be Morita equivalent to  $U_q \mathfrak{sl}(2)$  with  $q + q^{-1} = m - n$ . Usually, the representation theory of this centralizer – typically a quantum group or an algebra Morita equivalent to one – is easier to study than the representation theory of  $A$  or its scaling limit, a chiral algebra. In general, all these algebras are not semi-simple, and give rise to complicated indecomposable modules. The point is, in part, that these modules have, even for finite spin chains, properties which are closely related with those of the associated LCFT.

While the key observation in [10, 6] about the similarity between the algebraic properties of the lattice models and their logarithmic continuum limit is not entirely understood, it can be then interpreted in some cases at least in terms of quantum groups. Recent studies of centralizers of chiral algebras (Virasoro and W-algebras) in continuum logarithmic models have indeed unraveled a remarkable equivalence [13, 14, 15] between the representation theory and the fusion rules of the chiral algebras and certain quantum groups. It is exactly the same quantum groups that appear as centralizers in the lattice models, hence providing the link between finite size and scaling limit properties.

Turning now to bulk LCFTs, progress has been more modest. The problem is that, on the more abstract side, one now expects indecomposability under the left and right actions of the Virasoro algebras, leading to potentially very complicated modules which have proven too hard to study so far, except in some special cases. These include bulk logarithmic theories [16, 17] with W-algebra symmetries [18, 19], and WZW models on supergroups which, albeit very simple as far as LCFTs go, provide interesting lessons on the coupling of left and right sectors [20]. On the more concrete side, while it is possible to define and study lattice models whose continuum limit is a (bulk) LCFT, the underlying structures are also very difficult to get: symmetries are smaller, and the lattice algebras have much more complicated representation theory. Nevertheless, it looks possible to generalize the

approach in [10, 7] thanks to recent results about the centralizer  $\mathfrak{Z}_{JTL}$  of the Jones–Temperley–Lieb (JTL) algebra. This paper presents our first results in this direction for the  $\mathfrak{gl}(1|1)$  case. It is based on two technical companion papers. In the first one [21], we focussed on the symmetries of the spin chain, that is, the centralizer, in the alternating product of the  $\mathfrak{gl}(1|1)$  fundamental representation and its dual, of the JTL algebra. We proved that this centralizer is only a subalgebra of  $U_{\mathfrak{q}}s\ell(2)$  at  $\mathfrak{q} = i$  that we dubbed  $U_{\mathfrak{q}}^{\text{odd}}s\ell(2)$ . We then analyzed the continuum limit of the JTL algebra: using general arguments about the regularization of the stress-energy tensor, we identified families of JTL elements going over to the Virasoro generators  $L_n$  and  $\bar{L}_n$ . We also discussed the well known  $s\ell(2)$  symmetry of symplectic fermions from the lattice point of view, and showed that this symmetry, albeit present in the continuum limit, does not have a simple, useful analog on the lattice. In our second paper [22], we analyzed the decomposition of the spin chain over the Jones–Temperley–Lieb algebra, and obtained the full decomposition as a bimodule over  $U_{\mathfrak{q}}^{\text{odd}}s\ell(2)$  and  $JTL_N$ .

Equipped with these results, we can now explore to what extent the remarkable properties observed in the open case [10] carry over to the bulk case. Our conclusion is that the algebraic properties of the finite periodic spin chain and the bulk LCFT are again very similar. The crucial new ingredient is that the JTL algebra goes over, in the continuum limit, to a bigger algebra than  $\mathfrak{V}$ , the product of the left and right Virasoro algebras. This algebra – which we call *interchiral*, mixes the left and right moving sectors, and is generated, in the symplectic fermions case, by the additional field

$$S(z, \bar{z}) \equiv S_{\alpha\beta}\psi^\alpha(z)\bar{\psi}^\beta(\bar{z}), \quad S_{12} = S_{21} = 1, \quad S_{11} = S_{22} = 0, \quad (1.1)$$

with conformal weights  $(1, 1)$ . Most of the present work is devoted to identifying this algebra  $\mathcal{S}$ , studying its properties, and using it to provide a new analysis of the symplectic fermions LCFT. We believe that the concept of interchiral algebra will prove fundamental in the analysis of more complicated cases – in particular those at central charge  $c = 0$  – as will be discussed in forthcoming work.

The paper is organized as follows. We start in section 2 by a reminder of the main algebraic features, both on the lattice and in the scaling limit, for the open  $\mathfrak{gl}(1|1)$  spin chain (and its  $\mathfrak{gl}(n|n)$  generalizations). In section 3 we similarly remind the reader of the main algebraic properties of the closed  $\mathfrak{gl}(1|1)$  spin chain. The latter involves now a ‘periodic’ version of the Temperley–Lieb algebra which we call, following [10], the Jones–Temperley–Lieb algebra  $JTL_N$ , and a smaller symmetry algebra  $U_{\mathfrak{q}}^{\text{odd}}s\ell(2)$  which is, up to some trivial elements, its centralizer  $\mathfrak{Z}_{JTL}$ . The structure of the corresponding bimodule is recalled in Fig. 3. We also mention briefly the antiperiodic spin chain (where the  $\mathfrak{gl}(1|1)$  symmetry is broken), where one now deals with a twisted version  $JTL_N^{tw}$  of the Jones–Temperley–Lieb algebra, while the centralizer is just  $Us\ell(2)$  algebra. Section 4 is the first containing new results. We discuss there how, remarkably, the image of the  $JTL_N$  in the  $\mathfrak{gl}(1|1)$  spin chains is in fact isomorphic to (the image of) the enveloping algebra of a certain non-semisimple Lie algebra containing  $\mathfrak{sp}_{N-2}$  as its maximal semi-simple subalgebra. The semi-simple part of  $JTL_N$  (that is, after quotienting by the radical) is represented by  $U\mathfrak{sp}_{N-2}$ , providing a beautiful example of a classical Howe duality [23], for which we have a non semi-simple version in the full  $JTL_N$  case. This relationship with a Lie algebra provides us with a powerful tool to understand the scaling limit of  $JTL_N$ , which we call the **interchiral algebra**. This is discussed in section 5, which is, conceptually, the main section of this paper. For general models, we expect that the scaling limit of  $JTL_N$  will contain, in addition to the left and right Virasoro algebras, the modes of non chiral fields such as the

degenerate conformal field  $\Phi_{2,1} \times \bar{\Phi}_{2,1}$ . In the particular case of symplectic fermions, this field becomes the *interchiral field*  $S(z, \bar{z})$  from (1.1), and the scaling limit of  $JTL_N$  can be fully understood in terms of *bilinears in fermion modes*. This is discussed in subsections 5.1, 5.2 and – for the centralizer aspects – 5.3. The seemingly equivalent and more physical point of view where instead of bilinears in fermion modes we have bilinears in fermionic fields is discussed in subsection 5.4. Finally, we define the interchiral algebra proper in subsection 5.5 where it is denoted by  $\mathcal{S}$ . Subsection 5.6 extends the discussion to the antiperiodic model or the so-called “twisted sector”. In section 6 we discuss modules over the interchiral algebra in the LCFT and their relation with  $JTL_N$  modules on the lattice. We consider first simple modules in subsection 6.2 to 6.5. While on the finite lattice the simples are just fundamental representations of  $\mathfrak{sp}_{N-2}$ , in the limit simple modules over  $\mathcal{S}$  are identified with appropriate simple  $\mathfrak{sp}_\infty$ -modules. Of particular importance is our discussion of the vacuum module over the interchiral algebra in 6.3. In subsection 6.6 we discuss indecomposable modules over  $\mathcal{S}$  and their relation with the symplectic fermion theory. This involves an analysis of the Hilbert space of this theory as a module over the left-right Virasoro algebra  $\mathcal{V}(2)$  (which we believe is new) in subsubsection 6.6.1. We then show that this analysis agrees with the scaling limit of the bimodule over  $JTL_N$  and  $\mathfrak{Z}_{JTL}$ . A few conclusions – and pointers to subsequent developments – are given in the conclusion. Finally, several technical aspects are addressed in appendices.

## 1.1 Notations

To help the reader navigate through this long paper, we provide a partial list of notations (consistent with all other papers in the series)

$TL_N$  — the (ordinary) Temperley–Lieb algebra,

$T_N^a$  — the periodic Temperley–Lieb algebra with the translation  $u$ , or the algebra of affine diagrams,

$JTL_N$  — the Jones–Temperley–Lieb algebra,

$\mathfrak{Z}_{JTL}$  — the centralizer of  $JTL_N$ ,

$\pi_{\mathfrak{gl}}$  — the spin-chain representation of  $JTL_N$ ,

$\rho_{\mathfrak{gl}}$  — the spin-chain representation of the quantum group  $U_q \mathfrak{sl}(2)$ ,

$e, f$  — the renormalized powers,

$X_{1,n}$  — the simple  $U_i \mathfrak{sl}(2)$ -modules,

$P_{1,n}$  — the projective  $U_i \mathfrak{sl}(2)$ -modules,

$X_n$  — the simple  $U_i^{\text{odd}} \mathfrak{sl}(2)$ - and  $\mathfrak{Z}_{JTL}$ -modules,

$T_n$  — the indecomposable summands in spin-chain decomposition over the centralizer  $\mathfrak{Z}_{JTL}$ ,

$\mathcal{W}_j$  — the standard modules over  $TL_N$ ,

$\mathcal{P}_j$  — the projective modules over  $TL_N$ ,

$\mathcal{L}_{j,(-1)^{j+1}}$  — the simple modules over  $JTL_N$  for which we also use the notation  $(d_j^0)$ ,  
 $\mathcal{W}_{j,(-1)^{j+1}}$  — the standard modules over  $JTL_N$ ,  
 $\widehat{\mathcal{P}}_j$  — the indecomposable summands in spin-chain decomposition over  $JTL_N$ ,  
 $\mathfrak{S}_N$  — a Lie algebra introduced in Sec. 4.3.1,  
 $\mathfrak{S}'_N = \mathfrak{S}_N \oplus \mathbb{C}\mathbf{1}$  — a central extension of  $\mathfrak{S}_N$ ,  
 $\mathfrak{gl}_\infty$  — the Lie algebra of infinite matrices with finite number of non-zero elements, see Sec. 5.1.2,  
 $\mathfrak{gl}'_\infty$  — a central extension of the Lie algebra  $\mathfrak{gl}_\infty$ ,  
 $\mathfrak{sp}_\infty$  — the symplectic Lie algebra of infinite matrices,  
 $\mathfrak{S}_\infty$  — a Lie algebra of infinite matrices – the scaling limit of  $\mathfrak{S}'_N$  – introduced in Sec. 5.1.3,  
 $\mathfrak{S}$  — a Lie algebra of local operators introduced in Sec. 5.4.3,  
 $\mathcal{S}$  — the interchiral algebra, see Sec. 5.5,  
 $\mathcal{V}(2)$  — the left Virasoro algebra with  $c = -2$ ,  
 $\mathcal{V}(2)$  — the product of the left and right Virasoro algebras,  
 $\mathcal{X}_{j,1}$  — the simple Virasoro modules,  
 $\mathcal{P}_{n,1}$  — the staggered Virasoro modules,  
 $\mathcal{X}_j$  — a module over  $U\mathfrak{S}_\infty$  which is obtained in the scaling limit of the JTL modules  $\mathcal{L}_{j,(-1)^{j+1}}$ ,  
 $\mathcal{Y}_j$  — a module over  $U\mathfrak{S}_\infty$  which is obtained in the scaling limit of the JTL modules  $\widehat{\mathcal{P}}_j$ ,  
 $\mathcal{H}^+$  — the bosonic space of states in symplectic fermion theory,  
 $\mathcal{H}^-$  — the fermionic space of states in symplectic fermion theory.

## 2 A reminder of the open case

### 2.1 The open $\mathfrak{gl}(1|1)$ super-spin chain

The open  $\mathfrak{gl}(1|1)$  super-spin chain [10, 21] is a tensor product representation  $\mathcal{H}_N = \otimes_{j=1}^N \mathbb{C}^2$  of the Temperley–Lieb (TL) algebra of zero fugacity parameter  $m$ . We recall that the ordinary TL algebra denoted by  $TL_N(m)$  is generated by  $e_j$ , with  $1 \leq j \leq N-1$ , and has the defining relations

$$\begin{aligned}
e_j^2 &= m e_j, \\
e_j e_{j\pm 1} e_j &= e_j, \\
e_j e_k &= e_k e_j \quad (j \neq k, k \pm 1),
\end{aligned} \tag{2.1}$$

where  $m$  is a real parameter.

The representation space  $\mathcal{H}_N$  consists of  $N = 2L$  tensorands labelled  $j = 1, \dots, 2L$  with the fundamental representation of  $\mathfrak{gl}(1|1)$  on even sites and its dual on odd sites. The representation of each TL generator  $e_j$  is given by projectors on the  $\mathfrak{gl}(1|1)$ -invariant in the product of two neighbour tensorands

$$e_j^{\mathfrak{gl}} = (f_j + f_{j+1})(f_j^\dagger + f_{j+1}^\dagger), \quad 1 \leq j \leq 2L-1, \quad (2.2)$$

where we use a free fermion representation based on operators  $f_j$  and  $f_j^\dagger$  acting non-trivially only on  $j$ th tensorand and obeying

$$\{f_j, f_{j'}\} = 0, \quad \{f_j^\dagger, f_{j'}^\dagger\} = 0, \quad \{f_j, f_{j'}^\dagger\} = (-1)^j \delta_{jj'}, \quad (2.3)$$

where the minus sign for an odd  $j$  is due to presence of the dual representations of  $\mathfrak{gl}(1|1)$ . The generators  $e_j^{\mathfrak{gl}}$  provide then a representation of  $TL_{2L}(m=0)$  which is known to be faithful.

The representation space  $\mathcal{H}_{2L}$  is equipped with an inner product  $\langle \cdot, \cdot \rangle$  such that  $\langle f_j x, y \rangle = \langle x, f_j^\dagger y \rangle$  for any  $x, y \in \mathcal{H}_{2L}$ . We stress that the inner product is indefinite because of the sign factors in the relations (2.3). Then, the Hamiltonian operator

$$H_{\text{op.}} = - \sum_{j=1}^{2L-1} e_j^{\mathfrak{gl}},$$

with the ‘hamiltonian densities’  $e_j^{\mathfrak{gl}}$  defined in (2.2), is self-adjoint  $H_{\text{op.}} = H_{\text{op.}}^\dagger$  with respect to this inner product (actually, each  $e_j^{\mathfrak{gl}}$  is a self-adjoint operator). Its eigenvalues are real and the eigenvectors can easily be computed. As a consequence of the indefinite inner product, the self-adjoint Hamiltonian has non-trivial Jordan cells (of rank-two).

The open  $\mathfrak{gl}(1|1)$  spin-chain exhibits a large symmetry algebra dubbed  $\mathcal{A}_{1|1}$  in [10]. This algebra is the centralizer  $\mathfrak{Z}_{\text{TL}}$  of  $TL_N(0)$  and is generated by the identity and the five generators

$$F_{(1)} = \sum_{1 \leq j \leq N} f_j, \quad F_{(1)}^\dagger = \sum_{1 \leq j \leq N} f_j^\dagger, \quad (2.4)$$

$$F_{(2)} = \sum_{1 \leq j < j' \leq N} f_j f_{j'}, \quad F_{(2)}^\dagger = \sum_{1 \leq j < j' \leq N} f_{j'}^\dagger f_j^\dagger, \quad N = \sum_{1 \leq j \leq N} (-1)^j f_j^\dagger f_j. \quad (2.5)$$

We note that the formulas give, after trivial redefinitions, a representation of the (Lusztig) quantum group  $U_{\mathfrak{q}}sl(2)$  at  $\mathfrak{q} = i$ . The fermionic generators above, those with the subscript ‘(1)’, are from the nilpotent part and the bosonic ones form the  $sl(2)$  subalgebra in  $U_{\mathfrak{q}}sl(2)$ . It will be convenient in what follows to introduce slightly modified generators<sup>1</sup> of  $U_{\mathfrak{q}}sl(2)$

$$\mathfrak{h} = \frac{1}{2} \sum_{j=1}^N (-1)^j f_j^\dagger f_j - \frac{L}{2}, \quad \mathfrak{e} = \mathfrak{q}^{-1} \sum_{1 \leq j_1 < j_2 \leq N} f_{j_1}^\dagger f_{j_2}^\dagger, \quad \mathfrak{f} = \mathfrak{q} \sum_{1 \leq j_1 < j_2 \leq N} f_{j_1} f_{j_2}, \quad (2.6)$$

$$\mathfrak{K} = (-1)^{2\mathfrak{h}}, \quad \mathfrak{E} = \sum_{j=1}^N f_j^\dagger \mathfrak{K}, \quad \mathfrak{F} = \mathfrak{q}^{-1} \sum_{j=1}^N f_j \quad (2.7)$$

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<sup>1</sup>We dispense with the more correct notation  $\rho_{\mathfrak{gl}}$  used in [21].

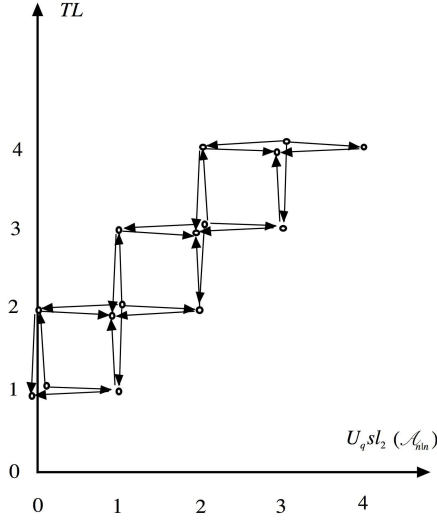


Figure 1: The structure of the open  $\mathfrak{gl}(1|1)$  spin-chain for  $N = 8$  sites, as a representation of the product  $TL_N \boxtimes U_i sl(2)$ . Some nodes occur twice and have been separated slightly for clarity.

obeying in particular

$$KEK^{-1} = -E, \quad KFK^{-1} = -F, \quad E^2 = F^2 = 0, \quad (2.8)$$

$$[h, e] = e, \quad [h, f] = -f, \quad [e, f] = 2h, \quad (2.9)$$

see more details in [21]. The  $sl(2)$  Cartan generator  $h$  is related to the total-spin operator  $S^z$  in the XX language, or the Fermion number  $N$  as

$$2h = S^z = N - L. \quad (2.10)$$

For our alternating chain, the values of  $S^z$  are integer since the number of sites is even, and thus  $h$  takes integer or half integer values.

## 2.2 Bimodule on a finite lattice

The decomposition of the open spin-chain as a bimodule over the pair  $(TL_N, \mathcal{A}_{1|1})$  of mutual centralizers is shown on Fig. 1 for the  $N = 8$  case and borrowed from [7]. The label  $j$  in the horizontal direction corresponds to the double value of the  $sl(2)$  spin  $n$  – the highest weight of an  $sl(2)$ -module. Each node with a Cartesian coordinate  $(j, k)$  in the bimodule diagram corresponds to a simple subquotient over the exterior product  $TL_N \boxtimes U_i sl(2)$  and arrows show the action of both algebras – the Temperley–Lieb  $TL_N$  acts in the vertical direction (preserving the coordinate  $j$ ), while  $U_i sl(2)$  acts in the horizontal way. Indecomposable projective  $TL_N$ -modules  $\mathcal{P}_k$  can be recovered by ignoring all the horizontal arrows, while tilting  $U_i sl(2)$ -modules  $\mathcal{P}_{1,j}$  (these are also projective) are obtained by ignoring all the vertical arrows. For more details, see [7, 22].

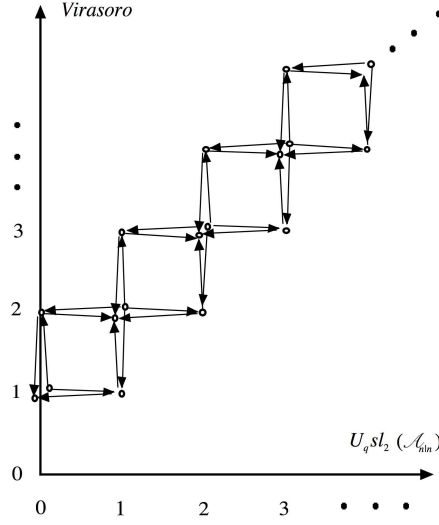


Figure 2: Space of states of the continuum theory as a representation of  $\mathcal{V}(2) \boxtimes U_i sl(2)$ .

### 2.3 Bimodule in the continuum limit

The crucial observation of [7] is that an identical bimodule structure, extending to arbitrarily high values of the  $sl(2)$  spin  $n = j/2$ , is present in the continuum limit. This is illustrated on Fig. 2, where the same comments as in the finite chain apply exactly, with the replacement of  $TL_N$  by the Virasoro algebra at central charge  $c = -2$  denoted by  $\mathcal{V}(2)$ .

It is useful here to comment Fig. 2 further. In the boundary symplectic fermion theory, the space of states decomposes as a direct sum of bosonic  $\mathcal{H}^+$  and fermionic  $\mathcal{H}^-$  sectors mixed by the fermionic part of the  $\mathfrak{gl}(1|1)$  algebra. We remind that the symplectic fermions form an  $sl(2)$  doublet [24] and the fermionic part of the  $\mathfrak{gl}(1|1)$  is formed by the two zero modes of the fermions. Each of the sectors is further decomposed as a direct sum of modules over the (exterior) product  $sl(2) \boxtimes \mathcal{V}(2)$  of the two commuting algebras,  $sl(2)$  and Virasoro, as (see also [25])

$$\mathcal{H}^+ = \bigoplus_{n \in \mathbb{N}_0} C_n \boxtimes \mathcal{P}_{2n+1,1}, \quad \mathcal{H}^- = \bigoplus_{n \in \mathbb{N} - \frac{1}{2}} C_n \boxtimes \mathcal{P}_{2n+1,1}, \quad (2.11)$$

where  $C_n$  denotes a  $(2n+1)$ -dimensional  $sl(2)$ -module of the (iso)spin  $n$ , and Virasoro modules  $\mathcal{P}_{n,1}$  are the so-called staggered modules introduced in [1]. Note that we use the term ‘isospin’ for  $sl(2)$ -modules on the CFT side in order to distinguish them from ones on the lattice, where the highest weight of an  $sl(2)$ -module we call in general by ‘spin’. In the open case, the  $sl(2)$  symmetry of the symplectic fermion theory [24] and the  $sl(2)$  symmetry in the full quantum group  $U_q sl(2)$  are actually coincident but this will not be true in the non-chiral case. Recall again that the value of  $S^z$  for the highest weight at (isospin)  $n$  is  $j = 2n$ , the label used in Fig. 2.



The staggered  $\mathcal{V}(2)$ -modules are indecomposable and have the following subquotient structure

$$\mathcal{P}_{2n+1,1} = \begin{array}{ccc} & \mathcal{X}_{2n+1,1} & \\ \swarrow & & \searrow \\ \mathcal{X}_{2n,1} & & \mathcal{X}_{2n+2,1} \\ \searrow & & \swarrow \\ & \mathcal{X}_{2n+1,1} & \end{array} \quad (2.12)$$

where  $\mathcal{X}_{2n+1,1}$  is the irreducible  $\mathcal{V}(2)$ -module with the conformal dimension  $\Delta_{2n+1,1} = n(2n+1)$ , with a non-negative integer or half-integer  $n$ . We note that a south-east arrow represents an action of negative Virasoro modes while a south-west arrow represent postive modes action. In the diagram (2.12), the staggered module is a ‘glueing’/extension of two indecomposable Kac modules which are highest-weight modules. The one in the top composed of  $\mathcal{X}_{2n+1,1}$  and  $\mathcal{X}_{2n+2,1}$  is the quotient of the Verma module with the weight  $\Delta_{2n+1,1}$  by the singular vector at the level  $2n+1$ , and the second Kac module in the bottom composed of  $\mathcal{X}_{2n,1}$  and  $\mathcal{X}_{2n+1,1}$  is a similar quotient (at the level  $2n$ ) of the Verma module with the weight  $\Delta_{2n,1}$ .

The different terms in  $\mathcal{H}^+$  and  $\mathcal{H}^-$  from (2.11) are in turn connected by the action of  $\mathfrak{gl}(1|1)$ , resulting into the diagram on Fig. 2, where each node is a product of an irreducible  $\mathcal{V}(2)$ -module and an irreducible  $s\ell(2)$ -module of dimension  $j+1$ . We see that the symmetry algebra in this CFT, the centralizer of  $\mathcal{V}(2)$ , is the semi-direct product of the fermionic part of  $\mathfrak{gl}(1|1)$  and the enveloping algebra  $U_q s\ell(2)$ . This centralizer turns out to coincide with (a representation of) the quantum group  $U_q s\ell(2)$  at  $q=i$ , which we also denote by  $\mathcal{A}_{1|1}$ , as on a finite spin-chain.

## 2.4 A note on other spin-chains

Note that while we discuss only  $\mathfrak{gl}(1|1)$  spin-chains here, a diagram identical to Fig. 2 describes the continuum limit of alternating  $\mathfrak{gl}(n|n)$  spin-chains. We recall that these spin-chains are defined in a similar fashion by introducing projections on  $\mathfrak{gl}(n|n)$ -invariants [26]. They give also faithful representations of  $TL_N(0)$  where the horizontal action in the bimodule diagram is due to the symmetry algebra being now  $\mathcal{A}_{n|n}$ , an algebra which is Morita equivalent to  $\mathcal{A}_{1|1}$  [10].

A similar analysis for other cases – representations of  $TL_N(m)$  for other values of  $m$  – such as the alternating  $\mathfrak{gl}(2|1)$  spin-chain or the XXZ spin-chains with  $U_q s\ell(2)$  symmetry at different roots of unity show that the lattice bimodule structure can be used along the same lines to infer properties of all known boundary logarithmic CFTs. In particular, the staggered Virasoro modules for different central charges abstractly discussed in [3] or [4] can quickly be recovered in this fashion, at least their subquotient structure can be deduced from the bimodules. This opens in particular the way to measuring [27, 28] indecomposability parameters (also called  $\beta$  invariants [12]) characterising Virasoro-module structure completely, or to computing fusion rules using an induction procedure [10, 7, 29].

These similarities in subquotient structures and fusion rules indicate an equivalence between corresponding tensor categories. Braided tensor structure or fusion data on the lattice part is given by an induction bi-functor associated with two lattices of arbitrary sizes joined to each other. Rigorously, this bi-functor gives a braided tensor structure only in the infinite size limit, where a construction of inductive limits of the finite categories of modules over the Temperley–Lieb algebras is required. We note that a systematic way to construct these limits is based on the spin-chain bimodules. Then, the inductive limits should be compared with braided tensor categories of modules over the Virasoro

algebras. We believe that a direct construction of centralizers of the Virasoro algebras in the LCFTs would give the desired equivalence.

This equivalence can be established by direct calculation in the case of symplectic fermions, where the continuum limit can be explicitly carried out (for steps in this direction, see [21]). For other cases however, such a calculation seems completely out of reach since models are only Bethe ansatz solvable, and precious little is known about their continuum limits, apart from critical exponents and some indecomposable modules. The equivalence thus remains a very reasonable conjecture, which will take considerably more work – in particular, in defining the continuum or scaling limit – before being rigorously established. It can nevertheless be easily understood if one recognizes the similar role played by the quantum group  $U_q\mathfrak{sl}(2)$  both on the lattice and in the continuum [15, 30].

Meanwhile, our strategy consists in postulating that similar equivalences between the lattice and continuum models are present in the bulk, or non-chiral, case as well. We shall then study the case of closed  $\mathfrak{gl}(1|1)$  spin-chains in detail, to draw lessons that we will then apply to more complicated – and physically interesting – models.

### 3 A reminder of the algebraic aspects in the closed case

#### 3.1 The Jones–Temperley–Lieb algebra $JTL_N(0)$ .

A closed periodic spin-chain can be obtained by adding a last generator  $e_N^{\mathfrak{gl}}$  defined using formula (2.2) but this time identifying the labels modulo  $N$ , in particular  $f_{N+1} \equiv f_1$ ,  $f_{N+1}^\dagger \equiv f_1^\dagger$ :

$$e_N^{\mathfrak{gl}} = (f_N + f_1)(f_N^\dagger + f_1^\dagger). \quad (3.1)$$

The generators  $e_j^{\mathfrak{gl}}$ , with  $1 \leq j \leq N$ , satisfy the same relations (2.1) but now the indices are interpreted modulo  $N$ .

The relations (2.1) (defined modulo  $N$ ) define an infinite dimensional associative algebra denoted in our companion papers by  $TL_N^a$  [31, 32]. The algebra  $TL_N^a$  is known in the physics literature as the periodic Temperley–Lieb algebra [34, 33]. In fact, the formulas for the generators in (2.2) and (3.1) lead to many more relations; as a result, the periodic  $\mathfrak{gl}(1|1)$  spin-chain provides a non-faithful representation of  $TL_N^a$ . Moreover, it provides a non-faithful representation of a finite-dimensional quotient – called *the Jones–Temperley–Lieb algebra*  $JTL_N(m=0)$  – of a slightly bigger algebra enlarging  $TL_N^a$  by a translation operator  $u^2$ , see e.g. [22]. We denote this representation in what follows by  $\pi_{\mathfrak{gl}} : JTL_N(0) \rightarrow \text{End}_{\mathbb{C}}(\mathcal{H}_N)$ .

To define the algebra  $JTL_{2L}(m)$  (the following description is taken almost verbatim from [10]), we first introduce an operator  $u^2$  (with inverse  $u^{-2}$ ) which translates any state to the right by 2 sites (so as to be consistent with the distinction of two types of sites carrying dual representations). We thus have  $u^2 e_j u^{-2} = e_{j+2}$  and we also impose the relation  $u^{2L} = 1$ . We then define abstractly an algebra of diagrams as is customary for the ordinary Temperley–Lieb (TL) algebra, but this time on an annulus (or finite cylinder), in which a general basis element corresponds to a diagram of  $2L$  sites on the inner, and  $2L$  on the outer boundary; the sites are connected in pairs, but only configurations that can be represented using lines inside the annulus that do not cross are allowed. Multiplication is defined in a natural way on these diagrams, by joining an inner to an outer annulus, and removing the interior sites. We emphasize that whenever a closed loop is produced when diagrams are multiplied together, this loop must be replaced by a numerical factor  $m$  (as for the TL algebra), even for loops that wind around

the annulus, as well as for those that are homotopic to a point. We also impose that non-isotopic (in the annulus) diagrams connecting the same sites are identified. Then, the resulting algebra is finite-dimensional and is generated by the elements  $e_j$  and  $u^2$ , and they obey the relations (2.1) defined modulo  $N$ , which however are not a complete set of defining relations. We note that the numerical factor  $m$  for winding loops is not a consequence of the stated relations, but a separate assumption.

In what follows, we consider only the case  $m = 0$  and use the notation  $JTL_N \equiv JTL_N(0)$ .

The Hamiltonian operator  $H_{\text{per.}}$  in the periodic case will be denoted simply by  $H$  and is given by

$$H = - \sum_{j=1}^{2L} e_j^{\mathfrak{gl}}. \quad (3.2)$$

This operator is also self-adjoint  $H = H^\dagger$  with respect to the inner product, as in the open case described above. Its eigenvalues are also real and the eigenvectors can easily be computed. As a consequence of the indefinite inner product, the self-adjoint Hamiltonian has non-trivial Jordan cells of rank-two [21], see also Sec. 5.1 below.

### 3.2 The centralizer $\mathfrak{Z}_{JTL}$ of $JTL_N$

In the closed case, while the  $\mathfrak{gl}(1|1)$  symmetry generated by  $F_{(1)}$ ,  $F_{(1)}^\dagger$ , and  $N$  remains, the bosonic  $sl(2)$  generators  $F_{(2)}$  and  $F_{(2)}^\dagger$  defined in (2.5) do not commute with the action of  $JTL_N$  [21]. What remains of  $\mathcal{A}_{1|1}$  or  $U_{\mathfrak{q}}sl(2)$  is only the fermionic generators

$$\begin{aligned} F_{(2n+1)} &= \sum_{\substack{1 \leq j_1 < j_2 < \dots \\ \dots < j_{2n+1} \leq 2L}} f_{j_1} f_{j_2} \dots f_{j_{2n+1}}, \\ F_{(2n+1)}^\dagger &= \sum_{\substack{1 \leq j_1 < j_2 < \dots \\ \dots < j_{2n+1} \leq 2L}} f_{j_1}^\dagger f_{j_2}^\dagger \dots f_{j_{2n+1}}^\dagger, \end{aligned} \quad n \geq 0, \quad (3.3)$$

which, together with the bosonic operators  $N$  and  $F_{(2L)}$  and  $F_{(2L)}^\dagger$ , generate the centralizer  $\mathfrak{Z}_{JTL}$  of the representation of  $JTL_{2L}$ .

For further reference, we recall another and more convenient for us description of the centralizer in terms of standard  $U_{\mathfrak{q}}sl(2)$  generators. We introduce the generators

$$E_m \equiv \mathfrak{e}^m E \frac{K^2 + 1}{2}, \quad F_n \equiv \mathfrak{f}^n F \frac{K^2 + 1}{2}, \quad m, n \geq 0, \quad (3.4)$$

which are represented on the spin-chain following the expressions (2.6) and (2.7). They generate a subalgebra in  $U_{\mathfrak{q}}sl(2)$  which we denote as  $U_{\mathfrak{q}}^{\text{odd}}sl(2)$ . This subalgebra has the basis  $E_n F_m \mathfrak{h}^k K^l$ , with  $n, m, k \geq 0$  and  $0 \leq l \leq 3$ . The positive Borel subalgebra is generated by  $\mathfrak{h}$  and  $E_n$  while the negative subalgebra – by  $\mathfrak{h}$  and  $F_n$ , for  $n \geq 0$ .

The result [21] is then that the centralizer  $\mathfrak{Z}_{JTL}$  of the (representation  $\pi_{\mathfrak{gl}}$  of)  $JTL_{2L}$  on the alternating periodic  $\mathfrak{gl}(1|1)$  spin-chain  $\mathcal{H}_{2L}$  is the algebra generated by (the representation of)  $U_{\mathfrak{q}}^{\text{odd}}sl(2)$  and  $\mathfrak{f}^L$ ,  $\mathfrak{e}^L$ . The correspondence with the fermionic expressions (3.3) is  $F_n \sim F_{(2n+1)}$ ,  $E_n \sim F_{(2n+1)}^\dagger$ , and  $N \sim \mathfrak{h}$ , *etc.* For more details, see [21].

Representation theory of  $U_{\mathfrak{q}}^{\text{odd}}sl(2)$  was studied in our second paper [22] of this series. The indecomposable  $U_{\mathfrak{q}}^{\text{odd}}sl(2)$  modules  $\mathbb{T}_n$ , with integer  $1 \leq n \leq L$ , which appear in the decomposition

of  $\mathcal{H}_{2L}$  can be seen as restrictions of the well known projective  $U_q \mathfrak{sl}(2)$ -modules  $P_{1,n}$ . They are all indecomposable, with dimension  $4n$ , the same as the dimension of  $P_{1,n}$ . Their subquotient structure is given also in [22].

### 3.3 Bimodule over $JTL_N$ and $\mathfrak{Z}_{JTL}$

Recall that the representation  $\pi_{\mathfrak{gl}}$  of  $JTL_N$  is non-faithful (faithfulness aspects will be discussed in more detail in our next paper [35].) There is thus no evident direct way to get a decomposition of the periodic  $\mathfrak{gl}(1|1)$  spin-chain like in the open (faithful) case. For example, the general theory [36] of projective modules over a cellular algebra (which includes  $TL_N(m)$  and  $JTL_N(m)$  algebras), which can be applied for a faithful representation [35], is not directly useful here. Instead, an indirect strategy is needed, which was discussed in detail in [22], and which we do not recall in detail here, concentrating only on the essential aspects.

We first give a diagram describing the subquotient structure of the bimodule  $\mathcal{H}_{2L}$  over the pair  $(JTL_N, \mathfrak{Z}_{JTL})$ . The two commuting actions are presented in Fig. 3 where we show a direct sum of indecomposable spin-chain modules  $\widehat{\mathcal{P}}_j$  over  $JTL_N$ , and simple subquotients over  $JTL_N$  will be denoted below by  $(d_{|j|}^0)$ . The direct sum is depicted as a (horizontal) sequence of diagrams for  $\widehat{\mathcal{P}}_j$  from  $j = -L$  on the left to  $j = L$  on the right. Each node in the diagram is a simple subquotient over the product  $JTL_N \boxtimes U_q^{\text{odd}} \mathfrak{sl}(2)$ . The action of  $JTL_N$  is depicted by vertical arrows while the action of  $U_q^{\text{odd}} \mathfrak{sl}(2)$  is shown by dotted horizontal lines connecting different  $JTL_N$ -modules.

In the diagram on Fig. 3, the first horizontal layer (at the bottom) contains the space of ground states and it consists of four nodes, which are simple  $JTL_N$ -modules  $(d_1^0)$ , and dotted arrows mixing them. These nodes and arrows describe the indecomposable  $U_q^{\text{odd}} \mathfrak{sl}(2)$ -module  $T_1$ . The second layer contains eight nodes corresponding to  $(d_2^0)$  and the dotted arrows contribute to the indecomposable module  $T_2$ , etc. We emphasize that we do not draw long-range arrows representing action of the generators  $F_{>0}$  and  $E_{>0}$  in modules  $T_{n>1}$  in order to simplify diagrams but the arrows can be easily recovered using the subquotient structure of  $T_n$  given in [22] – for example, the second layer of the bimodule contains in addition four long arrows going from the node  $\diamond$  at  $j = \mp 1$  to the node  $\bullet$  at  $j = \pm 2$ , and from the node  $\bullet$  at  $j = \pm 2$  to the node  $\blacklozenge$  at  $j = \mp 1$ . With this comment about arrows in mind, the reader can compare complexity of this bimodule with the open-case bimodule in Fig. 1.

For any sector with non-zero  $j = S^z$ , the subquotient structure for  $\widehat{\mathcal{P}}_j$  is given in Fig. 4, on the right, while the tower for  $\widehat{\mathcal{P}}_0$  is presented on the left. All the towers are ended by the pair of simple subquotients  $(d_L^0)$ . We note that the two simple subquotients at each level of the ladders are isomorphic. The Hamiltonian  $H$  from (3.2) acts by Jordan blocks of rank 2 on each pair of isomorphic simple subquotients with one at the top (having only outgoing arrows) and the second subquotient in the socle of the module (having only ingoing arrows). The Jordan block structure is due to presence of zero fermionic modes in the Hamiltonian as it is observed in [21], see also Sec. 4.1.2.

We note that the  $JTL_N$ -modules  $\widehat{\mathcal{P}}_j$  in Fig. 3 are drawn in opposite direction ‘from bottom to top’ comparing to diagrams in Fig. 4, in order to the space of ground states to be in the bottom of the diagram.

Finally, we recall some information about the simple  $JTL_N$  modules, which we denote in [22] by  $\mathcal{L}_{j,(-1)^{j+1}}$  or more conveniently by  $(d_j^0)$  above. The number  $d_j^0$  is their dimension, obtained from the

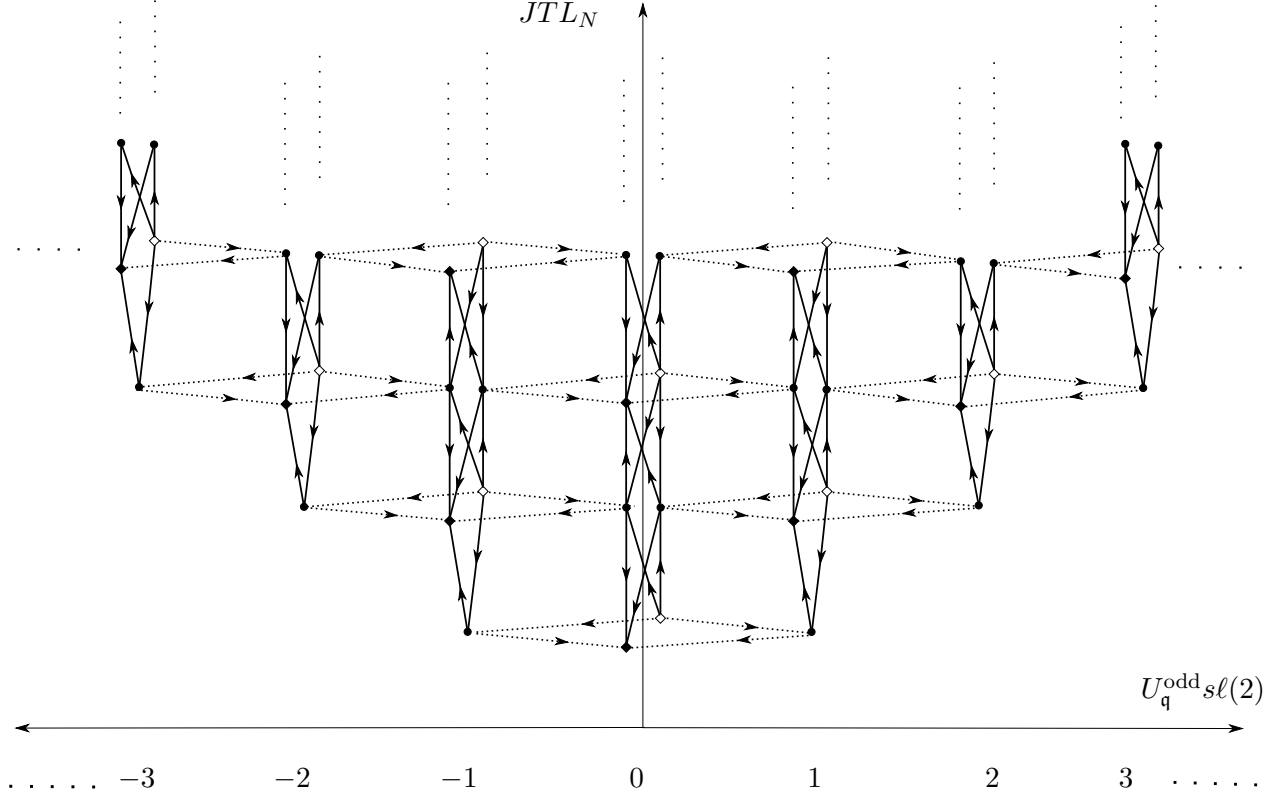


Figure 3: Bimodule over the pair  $(JTL_N, U_q^{\text{odd}}sl(2))$  of commuting algebras. The action of  $JTL_N$  is depicted by vertical arrows while the action of  $U_q^{\text{odd}}sl(2)$  is shown by dotted horizontal lines. Each label  $j$  in the horizontal axis corresponds to the sector for  $S^z$  and the label runs from  $-L$  on the left to  $j = L$  on the right. Each vertical tower above a label  $j$  is the diagram for  $\widehat{\mathcal{P}}_j$ . The first horizontal layer at the bottom contains four nodes ( $d_1^0$ ) and dotted arrows mixing them compose the  $U_q^{\text{odd}}sl(2)$ -module  $T_1$ . The second layer contains eight nodes ( $d_2^0$ ) and the dotted arrows depict the action in the indecomposable module  $T_2$ , *etc.* We suppress long-range arrows representing action of the generators  $F_{>0}$  and  $E_{>0}$  in order to simplify diagrams. For example, the second layer of the bimodule contains in addition four long arrows going from the node  $\diamond$  at  $j = \mp 1$  to the node  $\bullet$  at  $j = \pm 2$ , and from the node  $\bullet$  at  $j = \pm 2$  to the node  $\diamond$  at  $j = \mp 1$ .

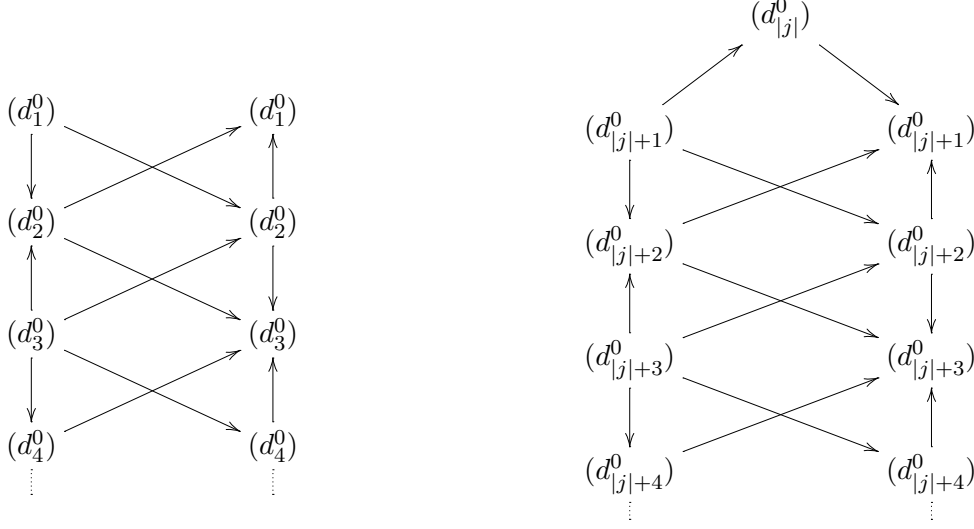


Figure 4: The two-strands structure of the spin-chain  $JTL_N$ -modules  $\widehat{\mathcal{P}}_j$  for  $j = 0$  on the left and  $j \neq 0$  on the right side. The towers are ended by the pair of  $(d_L^0)$ .

dimension of the spin-chain  $JTL_N$ -modules  $\widehat{\mathcal{P}}_j$  given by

$$\dim \widehat{\mathcal{P}}_j \equiv \widehat{d}_j = \binom{2L}{L+j} \quad (3.5)$$

which is just the dimension of the sector with  $S^z = j$  in the related XX spin chain with  $2L$  sites. We then have from the subquotient structure for  $\widehat{\mathcal{P}}_j$  in Fig. 4 that

$$\dim \mathcal{L}_{j,(-1)^{j+1}} \equiv d_j^0 = \sum_{j' \geq j} (-1)^{j'-j} d_{j'} \quad \text{with} \quad d_j = \binom{2L}{L+j} - \binom{2L}{L+j+1},$$

which can be written as

$$d_j^0 = \binom{2L-2}{L-j} - \binom{2L-2}{L-j-2}, \quad (3.6)$$

where  $0 \leq j \leq L$ . Note that  $d_0^0 = 0$  and we thus have that all simple modules over  $JTL_N$  that appear in this spin-chain are those  $\mathcal{L}_{j,(-1)^{j+1}}$  with  $1 \leq j \leq L$ , see also more details in our second paper [22]. We note also the obvious identities

$$\sum_{j=-L}^L \widehat{d}_j = \sum_{j=1}^L 4d_j^0 = 2^{2L}$$

which give the full dimension of the  $\mathfrak{gl}(1|1)$  spin-chain indeed.

It turns out that in this very degenerate case of  $JTL_N$  representations the  $\mathcal{L}_{j,(-1)^{j+1}}$  modules are also simple for the subalgebra  $TL_N$ . Of course, the difference with the open case is in the structure of indecomposable modules – the complexity of towers in Fig. 3 can be compared with the simpler ‘diamond’-type structure of  $TL$  modules in Fig. 1.

### 3.4 The antiperiodic spin-chain

We can also consider the alternating  $\mathfrak{gl}(1|1)$  spin-chain with antiperiodic conditions for the fermions, obtained by setting  $f_{2L+1}^{(\dagger)} = -f_1^{(\dagger)}$ . The generators  $e_j$ , for  $1 \leq j \leq 2L-1$ , have the same representation (2.2) while the last generator is then given by

$$e_{2L} = (f_{2L} - f_1)(f_{2L}^\dagger - f_1^\dagger), \quad (3.7)$$

to be compared with (3.1). This expression does not provide more a representation of the  $JTL_N$ : in the diagram language, non-contractible loops now should be replaced by the weight 2 (the dimension of the fundamental or its dual, instead of the superdimension in the periodic case); the relation  $u^N = (-1)^j$  should be also imposed in the sector with  $2j$  through-lines. We will call the corresponding algebra  $JTL_N^{tw}$ , see a precise definition in our second paper [22].

We emphasize that this antiperiodic  $\mathfrak{gl}(1|1)$  spin chain does not have  $\mathfrak{gl}(1|1)$  symmetry any longer. Instead, we have [21] that the centralizer of  $JTL_N^{tw}$  is generated by the  $sl(2)$  generators  $\mathbf{e}$  and  $\mathbf{f}$ , or  $F_{(2)}^\dagger$  and  $F_{(2)}$  defined in (2.5).

We then recall the bimodule structure [22] over the pair  $(JTL_N^{tw}, Usl(2))$ :

$$\mathcal{H}_N|_{JTL_N^{tw} \boxtimes Usl(2)} = \bigoplus_{j=0}^L \mathcal{L}_{j,(-1)^j} \boxtimes \mathbf{X}_{1,j+1} \quad (3.8)$$

where  $\mathbf{X}_{1,j}$  is the  $j$ -dimensional simple  $Usl(2)$ -module and  $\mathcal{L}_{j,(-1)^j}$  is a simple module over  $JTL_N^{tw}$ . The dimension of  $\mathcal{L}_{j,(-1)^j}$  is  $2d_{j+1}^0 + d_j^0 + d_{j+2}^0$  and is

$$\dim \mathcal{L}_{j,(-1)^j} = \binom{2L}{L+j} - \binom{2L}{L+j+2}. \quad (3.9)$$

Recall that the dimension of each  $S^z = j$  sector  $\mathcal{H}_{[j]}$  is  $\binom{2L}{L+j}$ . Then, the dimensions (3.9) correspond to single subtractions.

## 4 JTL algebra and Howe duality

We now give an interpretation of the simple modules over  $JTL_N$  and  $JTL_N^{tw}$  with the dimensions (3.6) and (3.9), respectively, from the point of view of representation theory of symplectic Lie algebras. We first recall a convenient notation for lattice fermions and then describe the representation theory of  $JTL_N$  in the periodic and anti-periodic spin chains in the context of (a non-semisimple version of) Howe duality.

### 4.1 A Lie algebra of fermion bilinears

Recall that the  $e_j$  generators of the  $JTL_N$  (or  $JTL_N^{tw}$ ) algebras in the (anti-)periodic spin-chains are linear combinations of bilinears in the fermions  $f_j$  and  $f_j^\dagger$ . For the periodic case, we have the representation

$$e_j^{\mathfrak{gl}} = (f_j + f_{j+1})(f_j^\dagger + f_{j+1}^\dagger), \quad 1 \leq j \leq 2L,$$

identifying the labels modulo  $N$ , in particular  $f_{N+1} \equiv f_1$ , and  $f_{N+1}^\dagger \equiv f_1^\dagger$  (see also expressions in (2.2) and (3.1)). For the anti-periodic model, we have a similar expression (3.7), where the fermions are

anti-periodic. It turns out that the commutators of these combinations of bilinears can be expressed again in fermionic bilinears, and of course belong to the JTL algebra. Obviously, they should generate a finite-dimensional Lie algebra. On the other hand, the spin-chain images of the JTL algebra contain many operators which are not bilinears. These non-bilinear operators are generated by the Lie algebra elements<sup>2</sup> because the  $e_j$  generators belong to the Lie algebra. We can thus expect that the full image of the JTL algebra can be described as an enveloping algebra of a Lie algebra (of course, it is also true for the open case discussed in Sec. 2 that the TL algebra, which is a subalgebra in the JTL algebra, is generated by a Lie algebra of fermionic bilinears). We begin our analysis with the anti-periodic spin-chain which is much simpler than the periodic one because of its semi-simplicity.

The centralizer of  $JTL_N^{tw}$  is generated as well by special bilinears in the fermions and these are the  $sl(2)$  generators  $\mathbf{e}$  and  $\mathbf{f}$ , or  $F_{(2)}^\dagger$  and  $F_{(2)}$  defined in (2.5). It is a well-known fact due to Howe [23] that, in the Clifford algebra  $\mathcal{C}_{4L}$ , the centralizing algebra of this image of  $Usl(2)$  – which is the enveloping algebra of the symplectic Lie algebra  $\mathfrak{sp}_2$  as well – is the enveloping algebra of  $\mathfrak{sp}_{2L}$ . Note that the dimensions (3.9) of the simple  $JTL_N^{tw}$ -modules coincide indeed with dimensions of fundamental representations of  $\mathfrak{sp}_{2L}$ . We can thus easily get the following theorem.

**Theorem 4.1.1.** *The image  $\pi_{\mathfrak{gl}}(JTL_N^{tw})$  of the twisted Jones–Temperley–Lieb algebra in the anti-periodic  $\mathfrak{gl}(1|1)$  spin-chain is isomorphic to the image of a representation of the enveloping algebra  $U\mathfrak{sp}_{2L}$  with all simple  $\mathfrak{sp}_{2L}$ -modules corresponding to one-column Young diagrams.*

Recall that in the anti-periodic model we have a double-centralizing property – see the corresponding semisimple  $(JTL_N^{tw}, U\mathfrak{sp}_2)$ -bimodule in Sec. 3.4. Then, the proof of this theorem is a simple consequence of the classical  $(\mathfrak{sp}(n), \mathfrak{sp}(m))$ -type Howe duality, where  $n = 2$  and  $m = 2L$  in our case.

We next describe explicitly the representation of  $\mathfrak{sp}_{2L}$  from Thm. 4.1.1, and introduce a more convenient fermion basis which we will use below in studying scaling limits.

#### 4.1.2 The Clifford algebra

The periodic spin chain admits the natural action of a Clifford algebra  $\mathcal{C}_{4L}$  with  $4L$  generators:  $\chi_p, \eta_p$  and their adjoints  $\chi_p^\dagger, \eta_p^\dagger$ , where we set  $p = m\pi/L$  and  $0 \leq m \leq L - 1$ . As operators, they are defined in App. A in (A3) and satisfy the following anti-commutation relations

$$\{\chi_p^\dagger, \chi_{p'}\} = \{\eta_p^\dagger, \eta_{p'}\} = \delta_{p,p'}, \quad \{\chi_p, \eta_{p'}\} = \{\chi_p^\dagger, \eta_{p'}^{(\dagger)}\} = \{\eta_p^\dagger, \chi_{p'}^{(\dagger)}\} = 0. \quad (4.1)$$

The Hamiltonian from (3.2) can then be written [21] in this Clifford algebra as

$$H = 2 \sum_{\substack{p=\epsilon \\ \text{step}=\epsilon}}^{\pi-\epsilon} \sin p (\chi_p^\dagger \chi_p - \eta_p^\dagger \eta_p) + 4\chi_0^\dagger \eta_0, \quad \epsilon = \frac{2\pi}{N}, \quad (4.2)$$

i.e., the  $\chi_p^{(\dagger)}$  and  $\eta_p^{(\dagger)}$  fermions are creation and annihilation operators that generate an eigenvector of  $H$  of the energy  $2\sin(p)$  and momentum  $p$ . We note that  $H$  is non-diagonalizable and its off-diagonal part  $\chi_0^\dagger \eta_0$  generates Jordan cells of rank 2. The creation operators generate all root- and eigen-vectors

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<sup>2</sup>Note that the translation generator  $u^2$  action in our spin-chains can be essentially expressed by products of  $e_j$ 's, see more concrete statements in [21, 22].



of  $H$  from the space of ground states. The latter has the following structure:

$$\begin{array}{ccc}
 & \omega & \\
 \chi_0^\dagger \swarrow & & \searrow -\eta_0 \\
 \phi^2 & & \phi^1 \\
 \eta_0 \searrow & H \downarrow & \swarrow \chi_0^\dagger \\
 & \Omega &
 \end{array} \tag{4.3}$$

where the two bosonic states – the vacuum  $\Omega$  and the state  $\omega$  – have  $S^z = 0$  and form a two-dimensional Jordan cell for the lowest eigenvalue for  $H$ , while the two fermionic states  $\phi^2$  and  $\phi^1$  belong to the sectors with  $S^z = +1$  and  $S^z = -1$ , respectively. We also showed in (4.3) the action of the quantum-group generators  $F \sim \eta_0$  and  $E \sim \chi_0^\dagger$ .

All the excitations over the ground states  $\omega$  and  $\Omega$  are thus generated by the free action of the fermionic creation modes which are  $\chi_p^\dagger$  and  $\eta_p$ , for  $p \in (0, \pi)$ , and  $\eta_0^\dagger$  and  $\chi_0$  (see precise definitions of these operators in (A3)). The annihilation modes acts as

$$\chi_p \Omega = \eta_p^\dagger \Omega = \chi_0^\dagger \Omega = \eta_0 \Omega = 0, \quad p \in (0, \pi). \tag{4.4}$$

For the antiperiodic spin-chains, we have also the action of a Clifford algebra with  $4L$  generators  $\chi_p$ ,  $\eta_p$  and adjoint ones  $\chi_p^\dagger$ ,  $\eta_p^\dagger$ , where now the momenta run over a different set:  $p = (m - \frac{1}{2})\pi/L$  and  $1 \leq m \leq L$ . In this case, there are no zero modes, and in particular the ground state of the Hamiltonian with anti-periodic conditions is non-degenerate [21].

#### 4.1.3 A special basis in JTL algebras

We note the well known fact that the bilinears in the generators of the Clifford algebra  $\mathcal{C}_{4L}$  introduced in (4.1) that commute with the operator  $S^z$  (or with the fermion number operator) give a basis in the Lie algebra  $\mathfrak{gl}_{2L}$ . Indeed, introducing the unit matrices  $\mathcal{E}_{m,n}$  with matrix elements unity at the corresponding  $(m, n)$ -th position and zero otherwise (a standard basis in  $\mathfrak{gl}_{2L}$ ), we can write them in terms of fermions as

$$\begin{aligned}
 \mathcal{E}_{m,n} &= \chi_p^\dagger \chi_q, & \mathcal{E}_{m,L+n} &= \chi_p^\dagger \bar{\eta}_q, \\
 \mathcal{E}_{L+m,n} &= \bar{\eta}_p^\dagger \chi_q, & \mathcal{E}_{L+m,L+n} &= \bar{\eta}_p^\dagger \bar{\eta}_q,
 \end{aligned} \quad 1 \leq m, n \leq L \tag{4.5}$$

where we set  $p = (m - \frac{1}{2})\pi/L$  and  $q = (n - \frac{1}{2})\pi/L$ , and we used also the notation  $\bar{\eta}_p = \eta_{\pi-p}$ ,  $\bar{\eta}_p^\dagger = \eta_{\pi-p}^\dagger$ . One can check using the relations (4.1) that the defining relations for  $\mathfrak{gl}_{2L}$ :

$$[\mathcal{E}_{m,n}, \mathcal{E}_{k,l}] = \delta_{n,k} \mathcal{E}_{m,l} - \delta_{m,l} \mathcal{E}_{k,n}, \quad 1 \leq m, n, k, l \leq 2L. \tag{4.6}$$

are satisfied. We then note that the linear combinations

$$\begin{aligned}
 \mathcal{A}_{m,n} &= \mathcal{E}_{m,n} - \mathcal{E}_{L+n,L+m}, & \mathcal{B}_{m,n} &= \mathcal{E}_{m,L+n} + \mathcal{E}_{L+m,n}, \\
 \mathcal{C}_{m,n} &= \mathcal{E}_{L+m,n} + \mathcal{E}_{L+n,m},
 \end{aligned} \quad 1 \leq m, n \leq L, \tag{4.7}$$

span a Lie subalgebra in  $\mathfrak{gl}_{2L}$  isomorphic to  $\mathfrak{sp}_{2L}$  (note that  $\mathcal{B}_{n,m} = \mathcal{B}_{m,n}$  and  $\mathcal{C}_{n,m} = \mathcal{C}_{m,n}$ ), and these combinations commute with the action of  $\mathfrak{sp}_2$  spanned by  $\mathbf{e}$ ,  $\mathbf{f}$ , and  $\mathbf{h}$ . Moreover, this Lie subalgebra is

centralized by the action of  $U\mathfrak{sl}(2)$  generated by the  $\mathfrak{sp}_2$ . The basis elements of  $\mathfrak{sp}_{2L}$  therefore generate the action of  $JTL_N^{tw}$ , as was stated in Thm. 4.1.1. We do not give here explicit expressions of  $e_j$ 's in terms of these basis elements of  $\mathfrak{sp}_{2L}$  but will give some formulas below. What is important to note now is that taking products of the  $\mathfrak{sp}_{2L}$  basis elements we obtain a special basis in the image of  $JTL_N^{tw}$  which is used later for taking the scaling limit of the JTL algebras.

Note finally that the  $\mathcal{A}_{n,n}$  generators are basis elements in the Cartan subalgebra of  $\mathfrak{sp}_{2L}$ . Then, the simple modules  $\mathcal{L}_{j,(-1)^j}$  over  $JTL_N^{tw}$  are highest-weight representations of  $\mathfrak{sp}_{2L}$  with the weights<sup>3</sup>  $(1, 1, \dots, 1, 0, \dots, 0)$ , which are sequences of 0's and 1's of length  $L$ , with  $j$  of 0's and  $0 \leq j \leq L$ . We thus obtain a decomposition (3.8) of the anti-periodic spin-chain with respect to the two commuting Lie algebras,  $\mathfrak{sp}_2$  and  $\mathfrak{sp}_{2L}$ , where the  $\mathfrak{sp}_2$  representation with  $j$  boxes corresponds to the  $j$ th fundamental (one-column) representation for  $\mathfrak{sp}_{2L}$ .

We then extend this classical Howe duality to the non-semisimple case which corresponds to the periodic model. The final result is given in Thm. 4.4 and Cor. 4.5 but first we discuss the semi-simple part of the action of the  $JTL_N$  algebra which parallels the previous discussion.

## 4.2 Semisimple part of $JTL_{2L}$ and Lie algebra $\mathfrak{sp}_{2L-2}$

For the periodic model, we introduce the standard basis for  $\mathfrak{gl}_{2L}$  as in (4.5), where now  $0 \leq m, n \leq L-1$  and  $p = m\pi/L$ , and  $q = n\pi/L$ . Note once again that we have zero modes  $\chi_0^{(\dagger)}$  and  $\eta_0^{(\dagger)}$  in this case, with the identification  $\bar{\eta}_0^{(\dagger)} = \eta_0^{(\dagger)}$ . Then, the linear combinations from (4.7), with  $1 \leq m, n \leq L-1$ , span a Lie subalgebra  $\mathfrak{sp}_{2L-2}$ . Note that we do not include zero modes because half of them do not commute with the symmetry algebra  $U_q^{\text{odd}}\mathfrak{sl}(2)$ . Recall that the  $U_q^{\text{odd}}\mathfrak{sl}(2)$  generators  $E_m$  and  $F_n$  are represented in the spin-chain by  $e^m E$  and  $f^n F$ , see (3.4) and note that  $K^2 = \mathbf{1}$  for even  $N$ . We can write [21] fermionic expression for these generators in the  $\chi$ - $\eta$  notation as

$$E_m = (-1)^{S^z} \sqrt{N} \left( \sum_{p=\epsilon}^{\pi-\epsilon} \eta_p^\dagger \chi_{\pi-p}^\dagger \right)^m \chi_0^\dagger, \quad F_n = -i\sqrt{N} \left( \sum_{p=\epsilon}^{\pi-\epsilon} \eta_p \chi_{\pi-p} \right)^n \eta_0. \quad (4.8)$$

It is then easy to check that the combinations of bilinears that give a basis in the  $\mathfrak{sp}_{2L-2}$  commute with this action of  $U_q^{\text{odd}}\mathfrak{sl}(2)$ . Moreover, we show below that the bilinears in fermions (4.7) are the only bilinears that do not contain zero modes and belong to the image of  $JTL_N$  under the representation  $\pi_{\mathfrak{gl}}$ . Put a bit differently, we show that the operators  $\mathcal{A}_{m,n}$ ,  $\mathcal{B}_{m,n}$ , and  $\mathcal{C}_{m,n}$  generate the semisimple part of the associative algebra  $\pi_{\mathfrak{gl}}(JTL_N)$ , while we will see later that similar combinations containing zero modes  $\chi_0^\dagger$  and  $\eta_0$  generate its Jacobson radical. Indeed, the dimensions (3.6) of the simple  $JTL_{2L}$ -modules coincide with dimensions of fundamental representations of  $\mathfrak{sp}_{2L-2}$ , as in the anti-periodic case but now for a smaller symplectic Lie algebra. This observation suggests the following lemma.

**Lemma 4.2.1.** *The image of the representation (4.7) of the enveloping algebra  $U\mathfrak{sp}_{2L-2}$  is isomorphic to the quotient  $\pi_{\mathfrak{gl}}(JTL_N)/\text{Rad}$ , where Rad is the Jacobson radical of the image of the Jones–Temperley–Lieb algebra in the periodic  $\mathfrak{gl}(1|1)$  spin-chain.*

*Proof.* We note that all simple modules  $\mathcal{L}_{j,(-1)^{j+1}}$  over  $JTL_N$  that appear in the spin-chain module  $\mathcal{H}_N$  appear also in the socle (maximum semisimple submodule) of  $\mathcal{H}_N$ , see Fig. 4. Then, in order to

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<sup>3</sup>Actually, for our choice of Cartan elements one should replace 1 by  $-1$  in order to obtain correct weights with respect to  $\mathcal{A}_{n,n}$ .

study the semisimple part of the action of  $JTL_N$ , it is enough to consider the action restricted on the socle of  $\mathcal{H}_N$ . Recall that simple modules over  $JTL_N$  are also simple modules over its subalgebra  $TL_N$ , see Sec. 3.3. Therefore, the centralizer of the action restricted on the socle – the intersection of the kernels of  $F$  and  $E$  in  $\mathcal{H}_N$  – is given by a representation of  $U\mathfrak{sp}_2$  realized by the divided powers of  $U_i\mathfrak{sl}(2)$ . Since the socle is freely generated from the vacuum state  $\Omega$  by the action of a Clifford algebra  $\mathcal{C}_{4L-4}$  with  $4L-4$  generators  $\chi_p^{(\dagger)}, \eta_p^{(\dagger)}$ , where  $p = m\pi/L$  and  $1 \leq m \leq L-1$ , we have a classical symplectic Howe duality [23] between  $\mathfrak{sp}_2$  and  $\mathfrak{sp}_{2L-2}$ . Indeed, the socle is a multiplicity-free semisimple bimodule over  $JTL_N$  and  $U\mathfrak{sp}_2$ , where both algebras are represented by appropriate bilinears in the generators of  $\mathcal{C}_{4L-4}$ . By Howe duality, we obtain that the action of  $JTL_N$  on the socle of  $\mathcal{H}_N$ , and thus its semisimple part on  $\mathcal{H}_N$ , is generated by the  $\mathfrak{sp}_{2L-2}$  action.  $\square$

In the open case, the simple modules are the same as for  $JTL_N$  in the periodic model. Therefore, we conclude that the semisimple part of  $TL_N$ , or the quotient  $TL_N/\text{Rad}$  by its Jacobson radical, is also generated by an  $\mathfrak{sp}_{N-2}$  action.

Using Lem. 4.2.1 and a correspondence between weights in the symplectic Howe duality [23, 37], we obtain the following corollary.

**Corollary 4.2.2.** *The simple modules  $\mathcal{L}_{j,(-1)^{j+1}}$ , with  $1 \leq j \leq L$ , over  $JTL_{2L}$  (or  $TL_{2L}$ ) are simple modules over the enveloping algebra for  $\mathfrak{sp}_{2L-2}$ . The dimensions (3.6) of all these simple modules correspond to all highest-weight representations of  $\mathfrak{sp}_{2L-2}$  of the weights  $(1, 1, \dots, 1, 0, \dots, 0)$ , which are sequences of 0's and 1's of length  $L-1$ , with  $j-1$  of 0's.*

We note once again that the generators  $\mathcal{A}_{n,n}$ , now with  $1 \leq n \leq L-1$ , span a basis in the Cartan subalgebra of the  $\mathfrak{sp}_{2L-2}$ . The diagonal part of the Hamiltonian  $H$  has a very simple expression in terms of these generators:

$$H = 2 \sum_{m=1}^{L-1} \sin(p) \mathcal{A}_{m,m} + 4 \mathcal{E}_{0,L}. \quad (4.9)$$

This allows us to describe  $JTL_N$  simple modules as highest-weight representations of  $\mathfrak{sp}_{2L-2}$  where highest-weight vectors play the role of charged vacua for the Hamiltonian. The vacuum state  $\Omega$  in the space of ground states coincides with the highest-weight vector of the unique  $\mathfrak{sp}_{2L-2}$  module of the weight  $(1, 1, \dots, 1, 1)$ , the next one corresponds to the weight  $(1, 1, \dots, 1, 0)$ , etc.

### 4.3 The image of $JTL_N$ as a Lie algebra representation

We introduce now special elements in  $JTL_N$  which span a subspace of all elements in  $JTL_N$  that are bilinear in the  $\chi$ - $\eta$  fermions. We begin with the definition of the Lie algebra  $\mathfrak{S}_N$ .

**Definition 4.3.1.** We define the Lie algebra  $\mathfrak{S}_N$  to be generated by  $\mathcal{A}_{m,n}, \mathcal{B}_{m,n}, \mathcal{C}_{m,n}$  from (4.7) and the unit matrices  $\mathcal{E}_{0,n}, \mathcal{E}_{0,L}, \mathcal{E}_{0,L+n}, \mathcal{E}_{m,L}$ , and  $\mathcal{E}_{L+m,L}$ , where  $1 \leq m, n \leq L-1$ . This Lie algebra can

be schematically depicted by matrices of the form (in the standard basis of  $\mathfrak{gl}_{2L}$ )

$$\mathfrak{S}_N : \begin{pmatrix} 0 & \times & \dots & \times & \times & \times & \dots & \times \\ 0 & & & & \times & & & \\ 0 & & \mathcal{A}_{m,n} & & \times & & \mathcal{B}_{m,n} & \\ 0 & & & & \times & & & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & & & & \times & & & \\ 0 & & \mathcal{C}_{m,n} & & \times & & & \\ 0 & & & & \times & & & \end{pmatrix}, \quad (4.10)$$

where the crosses  $\times$  stand for the corresponding elements  $\mathcal{E}_{0,n}$ ,  $\mathcal{E}_{0,L}$ ,  $\mathcal{E}_{0,L+n}$ ,  $\mathcal{E}_{m,L}$ , and  $\mathcal{E}_{L+m,L}$ .

Note that  $\mathfrak{S}_N$  is a non-semisimple Lie algebra and admits  $\mathfrak{sp}_{2L-2}$  spanned by  $\mathcal{A}_{m,n}$ ,  $\mathcal{B}_{m,n}$ , and  $\mathcal{C}_{m,n}$  as a Lie subalgebra. The dimension of  $\mathfrak{S}_N$  is

$$\dim \mathfrak{S}_N = \frac{(N-1)(N+2)}{2} - 1, \quad (4.11)$$

where we recall that  $\mathcal{B}$ - and  $\mathcal{C}$ -blocks are symmetric matrices. The Lie radical of  $\mathfrak{S}_N$  is generated by  $\mathcal{E}_{0,n}$ ,  $\mathcal{E}_{0,L}$ ,  $\mathcal{E}_{0,L+n}$ ,  $\mathcal{E}_{m,L}$ , and  $\mathcal{E}_{L+m,L}$ : these are the generators corresponding to the crosses  $\times$  in (4.10).

We next show that the representation of this Lie algebra of fermionic bilinears in the periodic spin-chain generates the image of  $JTL_N$ , proving the following theorem.

**Theorem 4.4.** *The image of the representation (4.5)-(4.7) of the enveloping algebra  $U\mathfrak{S}_N$  is isomorphic to the image  $\pi_{\mathfrak{gl}}(JTL_N)$  of the Jones–Temperley–Lieb algebra in the periodic  $\mathfrak{gl}(1|1)$  spin-chain.*

*Proof.* First, direct computation shows that the image of the representation (4.5)-(4.7) of the enveloping algebra  $U\mathfrak{S}_N$  commutes with the  $U_q^{\text{odd}}sl(2)$  action given in (4.8). We then show that the generators of  $\mathfrak{S}_N$  are the only bilinears in the  $\chi$ - $\eta$  fermions that commute with  $U_q^{\text{odd}}sl(2)$ . Let us for this introduce bilinears as in (4.7) but with the opposite sign and denote them by  $\mathcal{A}'_{m,n}$ ,  $\mathcal{B}'_{m,n}$ , and  $\mathcal{C}'_{m,n}$ , respectively. Together with the bilinears  $\mathcal{E}_{n,0}$  and  $\mathcal{E}_{L,n}$ , they belong to the complement of the vector space  $\mathfrak{S}_N$ . We then compute commutators of all these bilinears which are not in  $\mathfrak{S}_N$  with the generator  $E_1$ . It turns out that all these commutators are non-zero and linearly independent bilinears. So, there are no linear combinations among  $\mathcal{A}'_{m,n}$ ,  $\mathcal{B}'_{m,n}$ ,  $\mathcal{C}'_{m,n}$ ,  $\mathcal{E}_{n,0}$ , and  $\mathcal{E}_{L,n}$  that would commute with  $E_1$ . This proves the statement that  $\mathfrak{S}_N$  is the maximum Lie subalgebra of bilinears that commutes with  $U_q^{\text{odd}}sl(2)$ . We note then that the image of  $JTL_N$  is generated by  $e_j$ 's which are bilinears in the  $\chi$ - $\eta$  fermions and this image is thus contained in the image of  $U\mathfrak{S}_N$ .

To finish our proof, we need to show that the image of  $U\mathfrak{S}_N$  is not bigger than the image of  $JTL_N$ . To show this, we use a representation-theoretic approach. We recall the subquotient structure for each  $S^z = \pm j$  sector  $\widehat{\mathcal{P}}_j$  considered as a module over the centralizer of  $\mathfrak{J}_{JTL}$  which is isomorphic by the definition to the algebra  $\text{End}_{\mathfrak{J}_{JTL}}(\mathcal{H}_N)$ . The centralizer obviously contains  $\pi_{\mathfrak{gl}}(JTL_N)$ , as well as the image of  $U\mathfrak{S}_N$  of course, as a subalgebra. The opposite inclusion is not true as was shown in our second paper [22] and we thus can not rest on a double-centralizing argument.

The subquotient structure can be obtained using intertwining operators respecting  $\mathfrak{J}_{JTL}$  action. These are described in Thm. 3.4.4 in [22]. The only difference from the diagrams for  $JTL_N$  in Fig. 4

is that there are additional (‘long’) arrows mapping a top subquotient  $(d_j^0)$  (having only outgoing arrows) to  $(d_k^0)$  in the socle (having only ingoing arrows) whenever  $|j - k| \geq 4$  is an even number. We note these long arrows are not composites of any short arrows mapping from the top to the middle level<sup>4</sup>, and from the middle to the socle. It turns out that  $\mathfrak{S}_N$  generators correspond only to these short arrows and not to the long ones, and therefore there is no element from  $\mathfrak{S}_N$  represented by a long arrow. This can be shown using a direct calculation with the fermionic expressions (4.5)-(4.7). Due to Lem. 4.2.1, we need to analyze only the radical of  $\mathfrak{S}_N$  generated by  $\mathcal{E}_{0,n}$ ,  $\mathcal{E}_{0,L}$ ,  $\mathcal{E}_{0,L+n}$ ,  $\mathcal{E}_{m,L}$ , and  $\mathcal{E}_{L+m,L}$ . These are represented by bilinears in the Clifford algebra  $\mathcal{C}_{4L}$  generators. As was noted before,  $N - 2$  creation modes  $\chi_{p>0}^\dagger$  and  $\eta_{p>0}$  generate the bottom level – the intersection of the kernels of  $F$  and  $E$  in  $\mathcal{H}_N$  – from the vacuum state  $\Omega$ , and also the top level from one cyclic vector  $\omega$  which is involved with  $\Omega$  into a Jordan cell for the Hamiltonian  $H$ . Among the Clifford algebra generators, there are two – zero modes  $\eta_0$  and  $\chi_0^\dagger$  – proportional to  $F$  and  $EK^{-1}$ , respectively. These are the only generators mapping vectors from the top level to the middle level, and from the middle to the bottom level. Among the generators in the radical of  $\mathfrak{S}_N$ , there is only one,  $\mathcal{E}_{0,L}$ , which is the product of the two zero modes. The product maps the top to the bottom but commutes with the  $JTL_N$  action and thus maps a top subquotient  $(d_j^0)$  only to the bottom  $(d_j^0)$ . All other generators of the radical are bilinears in fermions containing only one of the zero modes and they thus map only by one level down. Therefore, they correspond to the short arrows in the diagram in Fig. 4. We conclude that the action of  $U\mathfrak{S}_N$  can not correspond to long arrows connecting the top and the bottom and which are not composites of any short arrows because any element of the radical in the image of  $U\mathfrak{S}_N$  is a linear combination of monomials in its generators.

We can thus conclude that the modules over the enveloping algebra of  $\mathfrak{S}_N$  in the spin-chain representation have the same subquotient structure as for the  $\pi_{\mathfrak{gl}}(JTL_N)$  and therefore their Jacobson radicals are isomorphic. In addition to the analysis on their semisimple parts given in Lem. 4.2.1 – both algebras have the same equivalence classes of simple modules – we also conclude that they are isomorphic as associative algebras. This finally proves the theorem.  $\square$

As a consequence of Thm. 4.4, we get the following corollary which we consider as a non-semisimple version of the (symplectic) Howe duality.

**Corollary 4.5.** *The centralizer  $\mathfrak{Z}_{U\mathfrak{S}_N}$  of the universal enveloping algebra  $U\mathfrak{S}_N$  in the periodic  $\mathfrak{gl}(1|1)$  spin-chain representation in (4.5)-(4.7) is given by  $\mathfrak{Z}_{JTL}$ , which is generated by the image of  $U_{\mathfrak{q}}^{odd}sl(2)$  in (4.8) and  $\mathfrak{f}^L$  and  $\mathfrak{e}^L$ .*

Note that in this case the “Howe-dual” algebra is not described as the enveloping of a Lie algebra of bilinears, in contrast to the anti-periodic case. It is rather a  $\mathbb{Z}_2$ -graded algebra due to presence of the  $\mathfrak{gl}(1|1)$  subalgebra. It is easier to show for the open case, where the  $JTL_N$  is the enveloping algebra of a non-semisimple Lie algebra (having maximum semisimple Lie subalgebra  $\mathfrak{sp}_{2L-2}$  and its radical is smaller than in the  $JTL_N$  case) and the action of its centralizer  $U_i sl(2)$  is generated by the semidirect product  $\mathfrak{sp}_2 \ltimes \mathfrak{gl}(1|1)$  of the Lie algebras  $\mathfrak{sp}_2$  and  $\mathfrak{gl}(1|1)$ .

## 4.6 Normal ordered basis

We introduce here a few formalities which will be useful to analyze the  $N \rightarrow \infty$  limits of our algebras. Recall that the excitations over the space of ground states are generated by the creation modes  $\chi_p^\dagger$  and

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<sup>4</sup>the level consisting of all nodes having both ingoing and outgoing arrows.

$\eta_p$ , with  $p \in (0, \pi)$ , while the annihilation modes are  $\chi_p$  and  $\eta_p^\dagger$ . We can thus introduce normal ordering prescriptions like  $:\chi_p \chi_{p'}^\dagger: \equiv -\chi_{p'}^\dagger \chi_p$ , *etc.* Then, a normal ordered basis in  $\mathfrak{gl}_{2L}$  is given by  $\mathcal{E}'_{m,n} = :\mathcal{E}_{m,n}:$ , where the unit matrices  $\mathcal{E}_{m,n}$ , for  $0 \leq m, n \leq 2L-1$ , are introduced above in (4.5). The normal ordering affects only one half of the Cartan elements:  $\mathcal{E}'_{n,n} = \mathcal{E}_{n,n}$  while  $\mathcal{E}'_{L+n,L+n} = \mathcal{E}_{L+n,L+n} - \mathbf{1}$ , where now  $0 \leq n \leq L-1$  and  $\mathbf{1}$  is the identity matrix. For the Lie subalgebra  $\mathfrak{sp}_{2L-2}$  we cannot choose such a basis because the identity is not in this subalgebra but we can extend it by the one-dimensional center spanned by the identity. So, we introduce the Lie algebra  $\mathfrak{sp}'_{2L-2} = \mathfrak{sp}_{2L-2} \oplus \mathbb{C}\mathbf{1}$  and then we choose the normal ordered basis in its Cartan subalgebra as  $\mathcal{A}'_{n,n} \equiv :\mathcal{A}_{n,n}: = \mathcal{E}'_{n,n} - \mathcal{E}'_{L+n,L+n} = \mathcal{A}_{n,n} + \mathbf{1}$ , see notations in (4.7). Note that the defining relations in the normally ordered basis are slightly changed. For the new basis in  $\mathfrak{gl}_{2L}$  we now have

$$[\mathcal{E}'_{m,L+n}, \mathcal{E}'_{L+k,l}] = \delta_{n,k} \mathcal{E}'_{m,l} - \delta_{m,l} \mathcal{E}'_{L+k,L+n} - \delta_{n,k} \delta_{m,l} \mathbf{1}, \quad (4.12)$$

while all other commutators have the same form as in the standard basis (4.6). Similarly, the commutators  $[\mathcal{B}_{m,n}, \mathcal{C}_{k,l}]$  have the central element term in the normally ordered basis in  $\mathfrak{sp}'_{2L-2}$  while other relations are not changed. Of course, we do the same things for the Lie algebra  $\mathfrak{S}_N$ : in order to work in the normally ordered basis we introduce the central extension  $\mathfrak{S}'_N = \mathfrak{S}_N \oplus \mathbb{C}\mathbf{1}$  where the commutators are slightly changed in the new basis by an identity contribution.

We also note the identity is expressed in the new basis in  $\mathfrak{gl}_{2L}$  as

$$\mathbf{1} = -\frac{1}{L-1} \sum_{n=0}^{2L-1} \mathcal{E}'_{n,n}. \quad (4.13)$$

Note finally that the weights of highest-weight  $\mathfrak{sp}'_{2L}$ -modules are now different due to a different choice of Cartan elements (or simple roots). For example, the vacuum irreducible module  $\mathcal{L}_{1,1}$  has now the weight  $(0, 0, \dots, 0)$ , the next one  $\mathcal{L}_{2,-1}$  corresponds to the weight  $(1, 0, \dots, 0)$ , *etc.*

## 5 Scaling limit, interchiral algebra, and local fields

While the structure of the symplectic fermion LCFT is fully consistent with the one of the  $\mathfrak{gl}(1|1)$  spin chain, there is an obvious difference due to the presence of the  $\mathfrak{sl}(2)$  symmetry in the continuum theory, which, in the periodic case, does not have a natural analog on the lattice. Investigation of more complicated models [35] suggests that the most productive way to think of this difference is to essentially forget the  $\mathfrak{sl}(2)$  symmetry, which seems to be an artifact from the point of view of the  $\mathfrak{gl}(1|1)$  theory. Rather, we believe that the bimodule structure of the lattice model does suggest to us the proper symmetries, and the proper way to analyze the scaling limit. Put otherwise, the good algebraic object that will organize the spectrum of most general logarithmic lattice models should be **the scaling limit of the Jones–Temperley–Lieb algebra**. Switching our point of view in this way leads to profound and maybe not so surprising conclusions, in particular that the organizing algebra of bulk LCFTs should contain non-chiral objects. Indeed, while the focus in the past [38] had been mostly to extract the stress-energy tensor modes from the Temperley–Lieb or Jones–Temperley–Lieb algebras, it is well known [26] that the scaling limit of some elements in  $JTL_N$  can lead to other physical observables corresponding to different bulk scaling fields. A very important example of such

a field is the energy operator<sup>5</sup> in the Potts model, associated with the staggered sum

$$\sum_{i=1}^N (-1)^i e_i \mapsto \int dx \Phi_{2,1} \times \bar{\Phi}_{21}(x, \tau = 0) \quad (5.1)$$

where the integral is taken over the circumference of a cylinder at constant imaginary-time  $\tau = 0$ . The field in (5.1) is the non chiral degenerate field with conformal weights  $h = \bar{h} = h_{2,1}$ ; later, we will denote it by  $S_{\text{cyl.}}(x, \tau)$ . Of course, the introduction of such fields in the organizing algebra of a LCFT requires discussion of objects which mix chiral and anti-chiral sectors. We shall in this section introduce the new concept of an **interchiral algebra**, and discuss its structure and role in the case of the closed  $\mathfrak{gl}(1|1)$  chains and the bulk symplectic fermions.

It is important to note that in the open case, the continuum limit of the Temperley–Lieb algebra  $TL_N$  only leads to the enveloping algebra of the Virasoro algebra, and does not involve other conformal fields. A related fact is, for instance, that the boundary energy operator at the ordinary transition in the Potts model coincides with the stress energy tensor [39]. There are probably cases for open spin chains where the algebra of hamiltonian densities will lead, in the continuum limit, to a bigger algebra than the Virasoro algebra – for instance, the super Virasoro algebra,  $W$ -algebras [40], *etc.*

From another, more formal point of view, recall we showed in Sec. 5 of our first paper [21] that the scaling limit of the JTL centralizer  $\mathfrak{Z}_{JTL}$  – an infinite dimensional representation of the  $U_q^{\text{odd}} \mathfrak{sl}(2)$  – gives an algebra of intertwining operators respecting the left and right Virasoro algebras. On the other hand, we also showed [21] that the centralizer of  $\mathfrak{V}(2)$  contains  $U_q^{\text{odd}} \mathfrak{sl}(2)$  but is a bigger algebra. It is thus reasonable to expect that the scaling limit of the Jones–Temperley–Lieb algebra is bigger than the non-chiral Virasoro algebra  $\mathfrak{V}(2)$ , and that the latter is only a proper subalgebra.

To make progress, we turn back to the scaling limit analysis of the  $\mathfrak{gl}(1|1)$  spin-chains discussed in [21] and extract an additional field that generates the full scaling limit of the JTL algebras. It turns out that the limit gives modes of a field of conformal dimension  $(1, 1)$ . We call this field the *interchiral field*  $S(z, \bar{z})$ ; it is expressed in terms of derivatives of the symplectic fermions as

$$S(z, \bar{z}) = S_{\alpha\beta} \psi^\alpha(z) \bar{\psi}^\beta(\bar{z}) \quad \text{with} \quad \psi^\alpha(z) = \partial \Phi^\alpha(z, \bar{z}), \quad \bar{\psi}^\alpha(\bar{z}) = \bar{\partial} \Phi^\alpha(z, \bar{z}), \quad (5.2)$$

where we introduced the symmetric form

$$S_{12} = S_{21} = 1, \quad S_{11} = S_{22} = 0. \quad (5.3)$$

We recall that the derivatives of the symplectic fermions have the mode decomposition [24]

$$\psi^\alpha(z) = \sum_{n \in \mathbb{Z}} \psi_n^\alpha z^{-n-1}, \quad \bar{\psi}^\alpha(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{\psi}_n^\alpha \bar{z}^{-n-1}, \quad (5.4)$$

and are primary fields of conformal dimensions  $(1, 0)$  and  $(0, 1)$ , respectively. The fermionic modes satisfy the anti-commutation relations

$$\{\psi_m^\alpha, \psi_{m'}^\beta\} = m J^{\alpha\beta} \delta_{m+m', 0}, \quad \alpha, \beta \in \{1, 2\}, \quad m, m' \in \mathbb{Z}, \quad (5.5)$$

with the symplectic form  $J^{12} = -J^{21} = 1$ , and the same formulas for the antichiral modes  $\bar{\psi}_n^\alpha$ , and  $\{\psi_m^\alpha, \bar{\psi}_{m'}^\beta\} = 0$ . We also introduce ‘constant’ modes  $\phi_0^2$  and  $\phi_0^1$  which are conjugate to the zero modes

$$\{\phi_0^1, \psi_0^2\} = i, \quad \{\phi_0^2, \psi_0^1\} = -i. \quad (5.6)$$

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<sup>5</sup>This is the field canonically coupled to the temperature.

## 5.1 The full scaling limit of the closed $\mathfrak{gl}(1|1)$ spin-chains

In this section, we recall the construction of the scaling limit [21] of the (anti)periodic spin-chain. We then use the Lie algebraic reformulation of the (images of)  $JTL_N$  and  $JTL_N^{tw}$  to construct the full scaling limit of these algebras. An essential ingredient in the general definition of the scaling limit is the low-lying eigenstates of the Hamiltonian  $H$ . In order to study the action of JTL elements on these eigenstates in the limit  $L \rightarrow \infty$  (recall  $N = 2L$ ) we first truncate each  $\mathcal{H}_{2L}$ , keeping only eigenspaces up to an energy level  $M$ , for each positive number  $M$ . Each such truncated space turns out to be finite-dimensional in the limit, *i.e.*, it depends on  $M$  but not  $L$ . Then, keeping matrix elements of JTL elements that correspond to the action only within these truncated spaces of “scaling” states, we obtain well-defined operators in the limit  $L \rightarrow \infty$ . The corresponding operators acting on all scaling states of the CFT can be finally obtained (if they exist) in the second limit  $M \rightarrow \infty$ . A bit more formally, *the scaling limit* denoted simply by ‘ $\mapsto$ ’ is defined as a limit over graded spaces of coinvariants with respect to smaller and smaller subalgebras in the creation modes algebra, see details in Sec. 4.3 from [21]. Meanwhile, in the case of our  $\mathfrak{gl}(1|1)$  spin-chains we can actually give a clearer definition of the scaling limit (of algebras), see App. C.

In this section, we consider mostly the periodic model and discuss briefly the anti-periodic case only at the end of Sec. 5.

### 5.1.1 The Clifford algebra scaling limit

We first recall [21] the scaling limit of the Clifford algebra  $\mathcal{C}_{4L}$  generators introduced in Sec. 4.1.2 – the  $\chi$  and  $\eta$  fermions – and then define a representation of  $\mathfrak{gl}_\infty$  which contains the full scaling limit of JTL. The lattice fermions appropriately rescaled coincides in the scaling limit with the symplectic-fermions modes:

$$\begin{aligned} \sqrt{m} \chi_p &\mapsto \psi_m^1, & \sqrt{m} \bar{\eta}_p^\dagger &\mapsto \psi_m^2, & \sqrt{m} \bar{\chi}_p &\mapsto \bar{\psi}_m^1, & -\sqrt{m} \eta_p^\dagger &\mapsto \bar{\psi}_m^2, \\ -\sqrt{m} \bar{\eta}_p &\mapsto \psi_{-m}^1, & \sqrt{m} \chi_p^\dagger &\mapsto \psi_{-m}^2, & \sqrt{m} \eta_p &\mapsto \bar{\psi}_{-m}^1, & \sqrt{m} \bar{\chi}_p^\dagger &\mapsto \bar{\psi}_{-m}^2, \end{aligned} \quad (5.7)$$

where we consider any finite integer  $m > 0$  and set  $p = m\pi/L$  and  $\bar{\eta}_p^{(\dagger)} = \eta_{\pi-p}^{(\dagger)}$ , and  $\bar{\chi}_p^{(\dagger)} = \chi_{\pi-p}^{(\dagger)}$ , while the scaling limit for zero modes is

$$\begin{aligned} \sqrt{L/\pi} \eta_0 &\mapsto \psi_0^1 = \bar{\psi}_0^1, & \sqrt{L/\pi} \chi_0^\dagger &\mapsto \psi_0^2 = \bar{\psi}_0^2, \\ \sqrt{\pi/L} \eta_0^\dagger &\mapsto i\phi_0^2, & \sqrt{\pi/L} \chi_0 &\mapsto -i\phi_0^1. \end{aligned} \quad (5.8)$$

Indeed, having this limit we obtain that the scaling limit of the Hamiltonian  $H = \sum_{j=1}^N e_j$  gives the zero mode of the stress energy tensor  $T(z) + \bar{T}(\bar{z})$  in the symplectic fermions theory:

$$\frac{L}{2\pi} \left( H + \frac{4L}{\pi} \right) \mapsto \sum_{m \in \mathbb{Z}} :\psi_{-m}^2 \psi_m^1: + (\psi \rightarrow \bar{\psi}) - \frac{c}{12} = L_0 + \bar{L}_0 - \frac{c}{12}, \quad (5.9)$$

where we use the fermionic normal ordering prescription introduced in Sec. 4.6 which takes the form [24]

$$:\psi_m^\alpha \psi_n^\beta: = \begin{cases} \psi_m^\alpha \psi_n^\beta & \text{for } m < 0, \\ -\psi_n^\beta \psi_m^\alpha & \text{for } m > 0. \end{cases} \quad (5.10)$$



We note that the fermionic operators act on the Hilbert space  $\mathcal{H}$  of scaling states, which are limits of low-energy eigenvectors<sup>6</sup> of the Hamiltonian  $H$ , and the ground subspace in  $\mathcal{H}$  has the same structure (4.3) as in the finite chain  $\mathcal{H}_N$ , where one should replace  $\chi_0^\dagger$  and  $\eta_0$  by the zero modes  $\psi_0^2$  and  $\psi_0^1$ , respectively. The module structure (over the infinite-dimensional Clifford algebra) on  $\mathcal{H}$  containing all the excitations over the ground states  $\omega$ ,  $\psi_0^2\omega$ ,  $\psi_0^1\omega$  and  $\psi_0^1\psi_0^2\omega = \Omega$  is thus generated by the free action of the creation modes  $\psi_n^{1,2}$  and  $\bar{\psi}_m^{1,2}$ , for integers  $n, m < 0$ . The annihilation modes, or positive modes, act on the vacuum states in the usual way.

We have also shown in our first paper [21] that the limit (5.7) allows one to obtain all left and right Virasoro modes  $L_n$  and  $\bar{L}_n$  in the symplectic fermions representation

$$L_n = \sum_{m \in \mathbb{Z}} :\psi_{n-m}^2 \psi_m^1:, \quad \bar{L}_n = \sum_{m \in \mathbb{Z}} :\bar{\psi}_{n-m}^2 \bar{\psi}_m^1:, \quad n \in \mathbb{Z}, \quad (5.11)$$

as the scaling limit of particular JTL elements, see also more details below in Sec. 5.4. A comment is necessary about the infinite sums in the definition of  $L_n$ 's and  $\bar{L}_n$ 's. Note that the Hilbert space  $\mathcal{H}$  has bi-grading by the pair  $(L_0, \bar{L}_0)$ , where each homogeneous or root subspace  $\mathcal{H}^{n,m}$  for positive  $n$  and  $m$  is spanned by states of the form  $\psi_{n_1}^{1,2} \dots \psi_{n_k}^{1,2} \bar{\psi}_{m_1}^{1,2} \dots \bar{\psi}_{m_l}^{1,2}$  acting on any ground state such that  $\sum_{i=1}^k n_i = -n$  and  $\sum_{j=1}^l m_j = -m$ . The space  $\mathcal{H}$  then can be described as the algebraic direct sum of its subspaces  $\mathcal{H}^{n,m}$  where the algebraic sum means that any state  $v \in \mathcal{H}$  can be written as  $v = \sum_{n,m} v^{(n,m)}$ , with  $v^{(n,m)} \in \mathcal{H}^{n,m}$ , and there exist positive integers  $N$  and  $M$  such that  $v^{(n,m)} = 0$  for any  $n > N$  or  $m > M$ . This is what we call *the finite-energy and finite-spin* (or simply *scaling*) *states space* and it is a dense subspace in the formal direct sum (or the direct product) of the root subspaces  $\mathcal{H}^{n,m}$ . Therefore, the formally infinite sums in (5.11) reduce actually to finite sums in our scaling states space  $\mathcal{H}$  and all the operators  $L_n$  and  $\bar{L}_n$  are in particular represented by bounded operators on  $\mathcal{H}$ . This is standard, and discussed for instance in [41].

### 5.1.2 The symplectic fermion representation of $\mathfrak{gl}_\infty$

We recall that the bilinears in the  $\chi$ - $\eta$  fermions give a standard basis  $\mathcal{E}_{m,n}$ , the usual unit matrices, in the Lie algebra  $\mathfrak{gl}_{2L}$ , see Sec. 4.2 for details. Due to the fact that low-lying eigenstates are generated by the Clifford generators with their momenta  $p$  close to 0 or  $\pi$  (in the large  $L$  limit), we keep in the scaling limit all bilinears in the fermions with any of these momenta. Note that basis elements (4.5) of  $\mathfrak{gl}_{2L}$  can be divided into 4 blocks with the  $(m, n)$  indices running from 0 to  $L-1$ . Because in our limit we keep only those indices close to 0 or to  $L$ , then each block is further divided into 4 infinite blocks in the scaling limit. This limit of  $\mathfrak{gl}_{2L}$  can be schematically described in terms of its standard generators as

$$\left[ \mathcal{E}_{m,n} = \chi_p^\dagger \chi_q \right]_{0 \leq p, q \leq L-1} \xrightarrow{L \rightarrow \infty} \begin{array}{ccccc} \chi_0^\dagger \chi_0 & \chi_0^\dagger \chi_\epsilon & \cdots & \chi_0^\dagger \bar{\chi}_{2\epsilon} & \chi_0^\dagger \bar{\chi}_\epsilon \\ \chi_\epsilon^\dagger \chi_0 & \chi_\epsilon^\dagger \chi_\epsilon & \cdots & \chi_\epsilon^\dagger \bar{\chi}_{2\epsilon} & \chi_\epsilon^\dagger \bar{\chi}_\epsilon \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{\chi}_{2\epsilon}^\dagger \chi_0 & \bar{\chi}_{2\epsilon}^\dagger \chi_\epsilon & \cdots & \bar{\chi}_{2\epsilon}^\dagger \bar{\chi}_{2\epsilon} & \bar{\chi}_{2\epsilon}^\dagger \bar{\chi}_\epsilon \\ \bar{\chi}_\epsilon^\dagger \chi_0 & \bar{\chi}_\epsilon^\dagger \chi_\epsilon & \cdots & \bar{\chi}_\epsilon^\dagger \bar{\chi}_{2\epsilon} & \bar{\chi}_\epsilon^\dagger \bar{\chi}_\epsilon \end{array} \quad (5.12)$$

for the left-top block in the matrix algebra, and similarly for other three blocks. Here, we set for the momenta variables  $p = m\epsilon$ ,  $q = n\epsilon$ , with  $\epsilon = \pi/L$ , and these momenta on the right-hand side are formal

<sup>6</sup>It is better actually to say ‘root-vectors’ because the Hamiltonian is not diagonalizable on  $\mathcal{H}_N$  and we have a basis where relations such as  $(H - \lambda \mathbf{1})^2 v = 0$ , for  $v \in \mathcal{H}_N$ , are satisfied.

variables – one should of course make substitutions (5.7) and (5.8). We emphasize that each item of the block on the right is an infinite unit matrix having identity at the corresponding position and zeros otherwise. These unit matrices (for all four blocks) define a representation of an infinite-dimensional Lie algebra  $\mathfrak{gl}_\infty$  on the space  $\mathcal{H}$  which we call the symplectic-fermion representation of  $\mathfrak{gl}_\infty$ .

A comment is necessary about the exact definition of the algebra  $\mathfrak{gl}_\infty$ . Any element of  $\mathfrak{gl}_\infty$  is an infinite matrix with a *finite* number of non-zero elements, *i.e.*, it is represented on  $\mathcal{H}$  by a finite sum of the generators – the unit matrices. The commutation relations in this algebra are given by corresponding limits of (4.6) and they correspond to usual basis and relations [42] in  $\mathfrak{gl}_\infty$  after appropriate rearranging rows and columns, see more precise statements in App. B.

We note that formally the Virasoro generators  $L_n$ 's and  $\bar{L}_n$ 's from (5.11) do not belong to such defined  $\mathfrak{gl}_\infty$  (recall there exist many versions of  $\mathfrak{gl}_\infty$  algebras [42]) but belong, for  $n \neq 0$ , to its completed version  $\overline{\mathfrak{gl}}_\infty$ , where there is now a possibly infinite number of non-zero elements, but the matrix still has a finite number of non zero diagonals. However, since we work in the scaling states space  $\mathcal{H}$  the  $L_n$ 's and  $\bar{L}_n$ 's, for  $n \neq 0$ , actually reduce to finite sums on  $\mathcal{H}$  and these operators thus belong to what we have called  $\mathfrak{gl}_\infty$ . The only problem is with  $L_0$  and  $\bar{L}_0$ , because of the well-known central anomaly that appears in CFT. In order to get a convenient algebra containing the  $L_0$  and  $\bar{L}_0$  operators we should actually take the scaling limit of the family of algebras  $\{\mathfrak{gl}_{2L}\}$  in their normally ordered basis  $\mathcal{E}'_{m,n}$  introduced in Sec. 4.6. Using then (5.7)-(5.8), we obtain finally the generators (after rescaling by  $\pm\sqrt{mn}$ ) of another infinite-dimensional Lie algebra we call  $\mathfrak{gl}'_\infty$  given by the following bilinears

$$\mathfrak{gl}'_\infty : \begin{pmatrix} \psi_0^2 \phi_0^1 & \psi_0^2 \psi_n^1 & \dots & \psi_0^2 \bar{\psi}_n^1 & \psi_0^2 \psi_0^1 & \psi_0^2 \psi_{-n}^1 & \dots & \psi_0^2 \bar{\psi}_{-n}^1 \\ \psi_{-m}^2 \phi_0^1 & \begin{bmatrix} \psi_{-m}^2 \psi_n^1 & \dots & \psi_{-m}^2 \bar{\psi}_n^1 \end{bmatrix} & & \psi_{-m}^2 \psi_0^1 & \begin{bmatrix} \psi_{-m}^2 \psi_{-n}^1 & \dots & \psi_{-m}^2 \bar{\psi}_{-n}^1 \end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{\psi}_{-m}^2 \phi_0^1 & \begin{bmatrix} \bar{\psi}_{-m}^2 \psi_n^1 & \dots & \bar{\psi}_{-m}^2 \bar{\psi}_n^1 \end{bmatrix} & & \bar{\psi}_{-m}^2 \psi_0^1 & \begin{bmatrix} \bar{\psi}_{-m}^2 \psi_{-n}^1 & \dots & \bar{\psi}_{-m}^2 \bar{\psi}_{-n}^1 \end{bmatrix} \\ \phi_0^2 \phi_0^1 & \phi_0^2 \psi_n^1 & \dots & \phi_0^2 \bar{\psi}_n^1 & \phi_0^2 \psi_0^1 & \psi_{-n}^1 \phi_0^2 & \dots & \bar{\psi}_{-n}^1 \phi_0^2 \\ \psi_m^2 \phi_0^1 & \begin{bmatrix} \psi_m^2 \psi_n^1 & \dots & \psi_m^2 \bar{\psi}_n^1 \end{bmatrix} & & \psi_m^2 \psi_0^1 & \begin{bmatrix} \psi_{-n}^1 \psi_m^2 & \dots & \bar{\psi}_{-n}^1 \psi_m^2 \end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{\psi}_m^2 \phi_0^1 & \begin{bmatrix} \bar{\psi}_m^2 \psi_n^1 & \dots & \bar{\psi}_m^2 \bar{\psi}_n^1 \end{bmatrix} & & \bar{\psi}_m^2 \psi_0^1 & \begin{bmatrix} \psi_{-n}^1 \bar{\psi}_m^2 & \dots & \bar{\psi}_{-n}^1 \bar{\psi}_m^2 \end{bmatrix} \end{pmatrix}, \quad (5.13)$$

which are now all normally ordered<sup>7</sup> and  $n$  and  $m$  are positive integers<sup>8</sup>. This Lie algebra turns out to be a central extension of the  $\mathfrak{gl}_\infty$ , *i.e.*, it is  $\mathfrak{gl}'_\infty = \mathfrak{gl}_\infty \oplus \mathbb{C}\mathbf{1}$ , as a vector space. The central element  $\mathbf{1}$  is obtained by the scaling limit of the identity matrix (4.13) in  $\mathfrak{gl}_{2L}$ , where one should rescale it as we did it for the Hamiltonian  $H$  in (5.9). Commutation relations in this algebra are obtained as limits of the relations (4.12) which fixes the central charge of  $\mathfrak{gl}'_\infty$ . The contribution of the central term in commutators of the new generators can be also obtained using the two-cocycle [42]  $c(\mathcal{E}'_{m,n}, \mathcal{E}'_{m',n'})$  and it coincides with the vacuum expectation of the commutator  $[\mathcal{E}'_{m,n}, \mathcal{E}'_{m',n'}]$  as it should. Now, all the left-right Virasoro operators on  $\mathcal{H}$  are embedded into the Lie algebra  $\mathfrak{gl}'_\infty$  including  $L_0$  and  $\bar{L}_0$ . We will actually work below with this Lie algebra  $\mathfrak{gl}'_\infty$  containing the Virasoro (acting on  $\mathcal{H}$ ) and not with  $\mathfrak{gl}_\infty$ .

<sup>7</sup>Note that some of these bilinears, those on the bottom or right half of the main diagonal, are not unit matrices.

<sup>8</sup>We assume for the block-matrices [...] that the modes of the generators (bilinears in the fermions) are growing in the directions from the corners of the four blocks to their centers.

### 5.1.3 The scaling limit of the JTL algebras, $\mathfrak{sp}_\infty$ , and $\mathfrak{S}_\infty$

The scaling limit of the (spin-chain representations of)  $JTL_N$  algebras can then be taken using the Lie algebraic description given in Thm. 4.4. It was shown there that the spin-chain representation of  $JTL_N$  is isomorphic to a representation of the enveloping algebra  $U\mathfrak{S}_N$ . We thus need only to take the scaling limit of the central extension  $\mathfrak{S}'_N = \mathfrak{S}_N \oplus \mathbb{C}\mathbf{1}$  introduced in Sec. 4.6 which is straightforward since there is an embedding of  $\mathfrak{S}'_N$  into the  $\mathfrak{gl}_{2L}$ , see Dfn. 4.10. In particular, its Lie subalgebra  $\mathfrak{sp}'_{2L-2}$  has the limit generated by the following bilinears

$$J_{\alpha\beta}:\psi_m^\alpha\psi_n^\beta:, \quad J_{\alpha\beta}:\bar{\psi}_m^\alpha\bar{\psi}_n^\beta:, \quad S_{\alpha\beta}\psi_m^\alpha\bar{\psi}_n^\beta, \quad m, n \in \mathbb{Z}/\{0\}, \quad (5.14)$$

where  $J_{\alpha\beta}$  is such that  $J_{\alpha\beta}J^{\beta\gamma} = \delta_\alpha^\gamma$  and we use the symplectic form  $J^{12} = -J^{21} = 1$ , and  $S_{\alpha\beta}$  is the symmetric form  $S_{12} = S_{21} = 1$ ,  $S_{11} = S_{22} = 0$ . We show below that the vector space spanned by finite linear combinations of these operators is closed under taking commutators. The infinite-dimensional Lie algebra they generate coincides with the central extension of the Lie algebra  $\mathfrak{sp}_\infty \subset \mathfrak{gl}_\infty$ .

Further, we introduce  $S_{m,n}$ ,  $T_{m,n}$ , and  $\bar{T}_{m,n}$  as the operators

$$S_{m,n} = S_{\alpha\beta}\psi_m^\alpha\bar{\psi}_n^\beta, \quad T_{m,n} = J_{\alpha\beta}\psi_m^\alpha\psi_n^\beta, \quad \bar{T}_{m,n} = J_{\alpha\beta}\bar{\psi}_m^\alpha\bar{\psi}_n^\beta, \quad \text{with } n, m \in \mathbb{Z}, \quad (5.15)$$

*i.e.*, as in (5.14) but without normal ordering and including combinations involving the zero fermionic modes  $\psi_0^1$  and  $\psi_0^2$  (note that there are no bilinears here involving conjugate modes  $\phi_0^2$  and  $\phi_0^1$ ). These operators together with the central term  $\mathbf{1}$  give now generators of the scaling limit of  $\mathfrak{S}'_N$  which we denote simply by  $\mathfrak{S}_\infty \subset \mathfrak{gl}'_\infty$ . We conclude that the corresponding action of  $U\mathfrak{S}_\infty$  is *the scaling limit* of the (spin-chain images of the) JTL algebras. We give in App. C a more formal construction of this scaling-limit algebra as a direct/inductive limit  $\varinjlim_L \pi_{\mathfrak{gl}}(JTL_{2L})$  of the finite-dimensional spin-chain representations  $\pi_{\mathfrak{gl}}$  of the JTL algebras.

### 5.1.4 The scaling limit of the JTL centralizers

It is possible with a little more formal work (see App. C) to define the *the scaling limit of the JTL centralizers*  $\mathfrak{Z}_{JTL}$ . It is an infinite dimensional algebra, which we identify with a quotient of  $U_q^{\text{odd}}sl(2)$ . It admits among irreducible representations only one-dimensional ones, see representation theory of  $U_q^{\text{odd}}sl(2)$  in [22]. We will denote this quotient also by  $U_q^{\text{odd}}sl(2)$  – this should not confuse the reader because we will specify the space it acts on. The fermionic expressions for the generators in the scaling limit of the centralizers were computed in our first paper [21] and they are

$$E_n = \left[ \sum_{m>0} \left( \frac{\psi_m^2\psi_{-m}^2}{m} - \frac{\bar{\psi}_m^2\bar{\psi}_{-m}^2}{m} \right) \right]^n \psi_0^2, \quad F_n = \left[ \sum_{m>0} \left( \frac{\psi_m^1\psi_{-m}^1}{m} - \frac{\bar{\psi}_m^1\bar{\psi}_{-m}^1}{m} \right) \right]^n \psi_0^1, \quad (5.16)$$

with the representation of the Cartan element  $\mathfrak{h}$  as

$$\mathfrak{h} = -i/2(\psi_0^2\phi_0^1 + \psi_0^1\phi_0^2) + \sum_{m>0} \frac{1}{m}(\psi_{-m}^2\psi_m^1 + \psi_{-m}^1\psi_m^2 + \bar{\psi}_{-m}^2\bar{\psi}_m^1 + \bar{\psi}_{-m}^1\bar{\psi}_m^2) \quad (5.17)$$

while the generator  $K = (-1)^{2\mathfrak{h}}$ . Note finally that the central term of  $\mathfrak{S}_\infty$  belongs also to the center of  $U_q^{\text{odd}}sl(2)$ .

## 5.2 Commutation relations in $\mathfrak{S}_\infty$

We next obtain commutation relations between the generators  $S_{n,m}$ ,  $T_{m,n}$  and  $\bar{T}_{m,n}$  of  $\mathfrak{S}_\infty$ . Using a straightforward calculation and identities like  $J_{\alpha\beta}J_{\gamma\delta}J^{\beta\gamma} = J_{\alpha\delta}$ , we first obtain

$$\begin{aligned} [T_{m,n}, T_{k,l}] &= [J_{\alpha\beta}\psi_m^\alpha\psi_n^\beta, J_{\gamma\delta}\psi_k^\gamma\psi_l^\delta] \\ &= J_{\alpha\beta}n(\delta_{n+k,0}\psi_m^\alpha\psi_l^\beta + \delta_{n+l,0}\psi_m^\alpha\psi_k^\beta) + J_{\alpha\beta}m(\delta_{m+k,0}\psi_l^\alpha\psi_n^\beta + \delta_{m+l,0}\psi_k^\alpha\psi_n^\beta) \end{aligned} \quad (5.18)$$

and a similar expression with  $\psi_n^\alpha \rightarrow \bar{\psi}_n^\alpha$ . We then have

$$[S_{m,n}, T_{k,l}] = S_{\alpha\beta}m(\delta_{k+m,0}\psi_l^\alpha\bar{\psi}_n^\beta + \delta_{m+l,0}\psi_k^\alpha\bar{\psi}_n^\beta), \quad (5.19)$$

$$[S_{m,n}, \bar{T}_{k,l}] = S_{\alpha\beta}n(\delta_{k+n,0}\psi_m^\alpha\bar{\psi}_l^\beta + \delta_{n+l,0}\psi_m^\alpha\bar{\psi}_k^\beta), \quad (5.20)$$

and the commutators

$$[S_{n,m}, S_{k,l}] = J_{\alpha\beta}(l\delta_{l+m,0}\psi_n^\alpha\psi_k^\beta + k\delta_{k+n,0}\bar{\psi}_m^\alpha\bar{\psi}_l^\beta) - 2kl\delta_{k+n,0}\delta_{l+m,0}, \quad (5.21)$$

where we use repeatedly the identities  $S_{\alpha\beta}J_{\gamma\delta}J^{\alpha\gamma} = S_{\beta\delta}$  and  $S_{\alpha\beta}J_{\gamma\delta}J^{\alpha\delta} = -S_{\beta\gamma}$ .

Using the definition of  $S_{n,m}$ ,  $T_{m,n}$  and  $\bar{T}_{m,n}$  given in (5.15), we see that the relations (5.18) and (5.21) together with (5.19) and (5.20) prove the following proposition which can be considered as an alternative definition of the  $\mathfrak{S}_\infty$  Lie algebra.

**Proposition 5.2.1.** *The Lie algebra  $\mathfrak{S}_\infty$  has  $S_{n,m}$ ,  $T_{k,l}$  and  $\bar{T}_{r,s}$ , with  $n, m, k, l, r, s \in \mathbb{Z}$ , as its basis elements, with the defining relations*

$$[S_{n,m}, S_{k,l}] = l\delta_{l+m,0}T_{n,k} + k\delta_{k+n,0}\bar{T}_{m,l} - 2kl\delta_{k+n,0}\delta_{l+m,0}, \quad (5.22)$$

$$[S_{m,n}, T_{k,l}] = m(\delta_{k+m,0}S_{l,n} + \delta_{m+l,0}S_{k,n}), \quad (5.23)$$

$$[S_{m,n}, \bar{T}_{k,l}] = n(\delta_{k+n,0}S_{m,l} + \delta_{n+l,0}S_{m,k}), \quad (5.24)$$

$$[T_{m,n}, T_{k,l}] = n(\delta_{n+k,0}T_{m,l} + \delta_{n+l,0}T_{m,k}) + m(\delta_{m+k,0}T_{l,n} + \delta_{m+l,0}T_{k,n}), \quad (5.25)$$

$$[\bar{T}_{m,n}, \bar{T}_{k,l}] = n(\delta_{n+k,0}\bar{T}_{m,l} + \delta_{n+l,0}\bar{T}_{m,k}) + m(\delta_{m+k,0}\bar{T}_{l,n} + \delta_{m+l,0}\bar{T}_{k,n}), \quad (5.26)$$

$$[T_{m,n}, \bar{T}_{k,l}] = 0. \quad (5.27)$$

We note that it is straightforward to check the Jacobi identities.

## 5.3 The centralizer of $\mathfrak{S}_\infty$

In order to study the symmetry algebra of the Lie algebra  $\mathfrak{S}_\infty$  we first note that using identities like

$$\left[\psi_m^2, \psi_{|m|}^1\psi_{-|m|}^1\right] = |m|\psi_m^1, \quad \left[\psi_m^1, \psi_{|m|}^2\psi_{-|m|}^2\right] = -|m|\psi_m^2, \quad m \in \mathbb{Z}, \quad (5.28)$$

it is straightforward to check that the generators (5.15) commute with the  $U_q^{\text{odd}}sl(2)$  action given in (5.16) and (5.17). Recall once again that  $JTL_N$  has a system of generators – those of the Lie algebra  $\mathfrak{S}_N$ , see Thm. 4.4 – that produce eigenstates of the Hamiltonian  $H$  and map eigenstates to other eigenstates. Taking the scaling limit, we keep only the generators corresponding to finite indexes  $(m, n)$ , as it is discussed in Sec. 5.1.2 above. Therefore, the statement about commuting actions is not very surprising because, since the centralizer of  $JTL_N$  acts on the eigenspaces for  $H$ , it should obviously be part of the symmetry algebra for the scaling limit of  $JTL_N$  or  $\mathfrak{S}_N$ .

On the other hand, the generators of  $\mathfrak{S}_\infty$  from (5.15) are the only bilinears in the symplectic fermions that commute with the action of  $U_q^{\text{odd}}sl(2)$ . To show this, we can repeat the beginning of the proof in Thm. 4.4, and change the range in the finite sums over fermionic bilinears to match our definition of  $\mathcal{H}$ : any state  $v \in \mathcal{H}$  can be written as a finite sum  $v = \sum_{n,m} v^{(n,m)}$ , with  $v^{(n,m)} \in \mathcal{H}^{n,m}$ , and there exist positive integers  $N$  and  $M$  such that  $v^{(n,m)} = 0$  for any  $n > N$  or  $m > M$ . This observation about fermionic bilinears that commute with  $U_q^{\text{odd}}sl(2)$  suggests that the centralizer of  $\mathfrak{S}_\infty$  might be given exactly by the representation of  $U_q^{\text{odd}}sl(2)$  on  $\mathcal{H}$ . To prove it by direct calculations seems hopeless to us but we can use the direct-limit construction of the Lie algebra  $\mathfrak{S}_\infty$  discussed in App. C. Indeed, the centralizer of the direct-limit algebra should be given by the inverse limit of the centralizers for each term  $\mathfrak{S}_N$ . We thus arrive at the following important proposition about the centralizer of the Lie algebra  $\mathfrak{S}_\infty$  of the fermionic bilinears.

**Proposition 5.3.1.** *The centralizer  $\mathfrak{Z}_{U\mathfrak{S}_\infty}$  of the enveloping algebra  $U\mathfrak{S}_\infty$  action (5.15) is given by the representation of  $U_q^{\text{odd}}sl(2)$  in (5.16) and (5.17).*

We discuss below in Sec. 6.1 the structure of the bimodule over the two commuting algebras,  $U_q^{\text{odd}}sl(2)$  and  $U\mathfrak{S}_\infty$ . In Sec. 6, we also describe simple and indecomposable modules over  $U\mathfrak{S}_\infty$  that appear in the space  $\mathcal{H}$  of scaling states but before, we consider a more physical interpretation of this algebra in terms of local fields.

## 5.4 Modes of local fields and the algebra $\mathfrak{S}$

While dealing with bilinears in fermionic modes is quite convenient and provides the most obvious approach to the scaling limit of JTL algebras, it is also important to understand what happens in more physical terms. From a (L)CFT point of view indeed, the natural objects are, for instance, not so much bilinears such as  $:\psi_{n-m}^2\psi_m^1:$ , but the modes of the stress energy tensor, which are (infinite) sums such as  $L_n = \sum_{m \in \mathbb{Z}} :\psi_{n-m}^2\psi_m^1:$ . Also, while Howe duality (in its non semi-simple version) gave us a powerful and new angle on  $JTL_N$  and its scaling limit, it is not likely that this will generalize to other models. It is thus crucial to be able to come up with an alternative, if less pleasant, approach to the scaling limit of JTL algebras.

### 5.4.1 Higher Hamiltonians

We begin by recalling the lattice analysis given in [21] where it was shown how to proceed from the  $JTL_N$  generators to get Virasoro modes in the non-chiral logarithmic conformal field theory of symplectic fermions: the combinations

$$H(n) = - \sum_{j=1}^N e^{-iqj} e_j^{\mathfrak{gl}}, \quad P(n) = \frac{i}{2} \sum_{j=1}^N e^{-iqj} [e_j^{\mathfrak{gl}}, e_{j+1}^{\mathfrak{gl}}], \quad q = \frac{n\pi}{L}, \quad (5.29)$$

of the  $JTL_N$  generators (recall that  $e_j^{\mathfrak{gl}}$  denotes the representation of  $e_j$  given in (2.2) and (3.1)) converge in the scaling limit as  $L \rightarrow \infty$  to the well-known symplectic fermions representation of the left and right Virasoro generators

$$\frac{L}{2\pi} H(n) \mapsto L_n + \bar{L}_{-n}, \quad \frac{L}{2\pi} P(n) \mapsto L_n - \bar{L}_{-n} \quad (5.30)$$

We note that the limit (5.30) of  $H(n)$  and  $P(n)$  is taken for finite  $n$ .

Here, we go further and consider also limits with  $n$  close to  $N/2$  (corresponding, physically, to alternating sum of JTL generators, related with the energy operator), or equivalently we consider  $H(L-k)$  and  $P(L-k)$  with  $L \rightarrow \infty$  and finite  $k$ . It turns out that this limit gives modes of a field of conformal dimension  $(1, 1)$ . To show this, we introduce a family of operators  $H_l(n)$  which are Fourier images of all the higher Hamiltonians:

$$H_l(n) = -\frac{1}{2}e^{-il\frac{\pi}{2}} \sum_{j=1}^N e^{-iqj} E_{j,l}, \quad q = \frac{n\pi}{L} \quad \text{and} \quad l \in \mathbb{N}, \quad n \in \mathbb{Z}, \quad (5.31)$$

where we used the following notation for multiple commutators of the  $JTL_N$ -generators  $e_j$

$$E_{j,l} = \left[ e_j, [e_{j+1}, \dots, e_{j+l-2}, [e_{j+l-1}, e_{j+l}] \dots] \right], \quad 1 \leq j \leq N. \quad (5.32)$$

We call these Fourier images generalized higher Hamiltonians. Note that  $P(n) = H_1(n)$  and we set  $H_0(n) = H(n)$ .

The fermionic expressions for the  $H_l(n)$ 's were obtained in our first paper [21] and it turns out that the  $H_l(n)$ 's can be written in the basis (4.10) of the Lie algebra  $\mathfrak{S}_N$ , for  $l \geq 0$  and  $0 \leq n \leq L-1$ , as

$$\begin{aligned} H_l(n) = & 2e^{iq\frac{l+1}{2}} \left( \sum_{m=1}^{L-n-1} \cos l(p + \frac{q}{2}) \sqrt{\sin(p) \sin(q+p)} \mathcal{A}_{m,m+n} \right. \\ & + \frac{1}{2} \sum_{m=1}^{n-1} \cos l(p - \frac{q}{2}) \sqrt{\sin(p) \sin(q-p)} (\mathcal{C}_{m,n-m} + (-1)^l \mathcal{B}_{L-m,L+m-n}) \\ & \left. + \cos \frac{lq}{2} \sqrt{\sin q} (\mathcal{E}_{0,n} + (-1)^l \mathcal{E}_{0,2L-n} + \mathcal{E}_{L+n,L} + (-1)^l \mathcal{E}_{L-n,L}) + 2\delta_{n,0}(1 + (-1)^l) \mathcal{E}_{0,L} \right), \quad (5.33) \end{aligned}$$

where we set  $p = m\pi/L$  and  $q = n\pi/L$  as usually. The expressions for negative values of  $n$  are given in (D4). Note that all zero modes  $H_l(0)$  and the lattice translation generator  $u^2$  have very simple expressions:

$$H_l(0) = 2 \sum_{m=1}^{L-1} \cos(lp) \sin(p) \mathcal{A}_{m,m} + 4(1 + (-1)^l) \mathcal{E}_{0,L} \quad (5.34)$$

and

$$u^2 = \exp \left( -\frac{2i\pi}{L} \sum_{n=1}^{L-1} n \mathcal{A}_{n,n} \right) \quad (5.35)$$

and they belong, for odd values of  $l$ , to the Cartan subalgebra of  $\mathfrak{sp}_{2L-2}$ .

We check then that the element  $H_0(L)$  is the energy operator indeed  $\sum_{j=1}^N (-1)^j e_j$  on the lattice (which differs from the Hamiltonian  $H$ ), which reads

$$\sum_{j=1}^N (-1)^j e_j = 2i \sum_{\substack{p=\epsilon \\ \text{step}=\epsilon}}^{\pi-\epsilon} \sin p (\chi_p^\dagger \eta_p + \eta_p^\dagger \chi_p). \quad (5.36)$$

The energy operator rescaled by the factor  $\frac{L}{2\pi i}$  has then the limit

$$\frac{L}{2\pi i} \sum_{j=1}^N (-1)^j e_j \mapsto \sum_{m \in \mathbb{Z}} (\psi_m^2 \bar{\psi}_m^1 + \psi_m^1 \bar{\psi}_m^2) \equiv S_0.$$

We denote this limit by  $S_0$  and show below that it corresponds to the zero mode of a primary field of dimension  $(1,1)$  but first show emergence of its higher modes  $S_n$ . We refer to our calculations in App. D where we obtain in the scaling limit for  $H_0(L-k)$  (keeping only the leading order in  $1/N$ ), with finite integer  $k$ ,

$$\frac{L}{2\pi i} H_0(L-k) \mapsto S_{\alpha\beta} \sum_{m \in \mathbb{Z}} \psi_m^\alpha \bar{\psi}_{m+k}^\beta \equiv S_k^{(0)}, \quad k \in \mathbb{Z}, \quad (5.37)$$

and the scaling limit for  $H_1(L-k)$  is

$$-\frac{L^2}{\pi^2} H_1(L-k) \mapsto S_{\alpha\beta} \sum_{m \in \mathbb{Z}} (2m+k) \psi_m^\alpha \bar{\psi}_{m+k}^\beta \equiv S_k^{(1)}, \quad k \in \mathbb{Z}, \quad (5.38)$$

where we introduce, for  $l \geq 0$  and  $k \in \mathbb{Z}$ , the operators

$$S_k^{(l)} = S_{\alpha\beta} \sum_{m \in \mathbb{Z}} (2m+k)^l \psi_m^\alpha \bar{\psi}_{m+k}^\beta. \quad (5.39)$$

The whole family of operators  $S_k^{(l)}$ , with  $l \geq 0$  and  $k \in \mathbb{Z}$ , can be extracted from the scaling limit of the operators  $H_l(L-k)$  keeping not only the leading terms in their formal expansion in  $1/L$  but also all sub-leading terms in the expansion. Put differently, we can formally expand  $H_l(L-k)$  as Laurent series in  $N$  where coefficients are fermionic bilinears of the form  $S_k^{(l)}$ . For simplicity, we refrain from discussing this any further here.

### 5.4.2 Fields

The foregoing family of operators  $S_k^{(l)}$ , with  $l \geq 0$  and  $k \in \mathbb{Z}$ , can be obtained as modes of a primary field of conformal dimension  $(1,1)$ , and of its descendants. We denote this primary field defined on the complex plane as

$$S_{\text{pl.}}(z, \bar{z}) = S_{\alpha\beta} \psi^\alpha(z) \bar{\psi}^\beta(\bar{z}), \quad (5.40)$$

where  $S_{\alpha\beta}$  is the symmetric form, *i.e.*,  $S_{12} = S_{21} = 1$ ,  $S_{11} = S_{22} = 0$ . We call this field the *interchiral* field; note that it is of course neither chiral nor anti-chiral.

Recall first the mode decomposition of the fermion fields

$$\psi^\alpha(z) = \sum_{n \in \mathbb{Z}} \psi_n^\alpha z^{-n-1}, \quad \bar{\psi}^\alpha(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{\psi}_n^\alpha \bar{z}^{-n-1}. \quad (5.41)$$

Then,  $S_{\text{pl.}}(z, \bar{z})$  has a formal expansion

$$S_{\text{pl.}}(z, \bar{z}) = S_{\alpha\beta} \sum_{n, m \in \mathbb{Z}} \psi_n^\alpha \bar{\psi}_m^\beta z^{-n-1} \bar{z}^{-m-1}. \quad (5.42)$$

Interpreting  $z$  and  $\bar{z}$  as independent complex coordinates allows one to extract  $\psi_n^\alpha \bar{\psi}_m^\beta$  by a double integration over  $z, \bar{z}$ . This is not too pleasant however, since we rather would like to deal with modes obtained by integration in the physical theory, where  $z, \bar{z}$  are not independent. Note also that in the Virasoro case, there is no  $\bar{z}$  we can integrate over, so there is no integration that can give rise to the single product  $\psi_n^\alpha \bar{\psi}_m^\beta$  starting from the stress energy tensor and its descendants.

To make contact with the hamiltonian formulation we used so far, it is now convenient to perform a conformal transformation  $z \rightarrow w(z)$ ,  $\bar{z} \rightarrow w^*(\bar{z})$  onto a cylinder of circumference  $L'$ , with  $w(z) = \frac{L'}{2\pi} \log(z)$ . We set  $w = \tau - ix$  and now  $w^* = \tau + ix$ , we then obtain

$$S_{\text{cyl.}}(x, \tau) = \left(\frac{2\pi}{L'}\right)^2 z \bar{z} S_{\text{pl.}}(z, \bar{z}) = \left(\frac{2\pi}{L'}\right)^2 S_{\alpha\beta} \sum_{n,m \in \mathbb{Z}} \psi_n^\alpha \bar{\psi}_m^\beta \exp\left[-\frac{2\pi}{L'}(\tau(m+n) + ix(m-n))\right]. \quad (5.43)$$

Introducing the mode expansion of the field  $S_{\text{cyl.}}(x, \tau)$  on the cylinder in the Heisenberg presentation, where the modes depend on the imaginary time  $\tau$ ,

$$S_{\text{cyl.}}(x, \tau) = \left(\frac{2\pi}{L'}\right)^2 \sum_{k \in \mathbb{Z}} S_k(\tau) e^{2\pi i k x / L'} = \left(\frac{2\pi}{L'}\right)^2 \sum_{k \in \mathbb{Z}} S_k(0) e^{-2\pi k w / L'}, \quad (5.44)$$

we then obtain

$$S_k(\tau) = \frac{L'}{(2\pi)^2} \int_0^{L'} dx e^{-2\pi i k x / L'} S_{\text{cyl.}}(x, \tau) \quad (5.45)$$

which is evaluated as

$$S_k(\tau) = S_{\alpha\beta} \sum_{n \in \mathbb{Z}} \psi_{n+k}^\alpha \bar{\psi}_n^\beta e^{-2\pi\tau(2n+k)/L'}. \quad (5.46)$$

Finally, we get the expansion

$$S_k(\tau) = \sum_{l \geq 0} \frac{(-2\pi\tau)^l}{l!(L')^l} S_{-k}^{(l)} \quad (5.47)$$

and in particular

$$S_k(0) = S_{-k}^{(0)}, \quad k \in \mathbb{Z}, \quad (5.48)$$

where  $S_k^{(l)}$  were introduced above in (5.39).

It is important to note that applying the zero mode  $S_0$  of the local field  $S_{\text{cyl.}}(x, \tau)$  on the vacuum state, we get then an infinite sum  $\sum_{m < 0} (\psi_m^2 \bar{\psi}_m^1 + \psi_m^1 \bar{\psi}_m^2) \Omega$ . All other modes  $S_k$  have similar formally divergent action. These formal divergences are not very surprising as a local operator in general is able to excite states of arbitrarily high energy (the spin is fixed in the action of our local operators). Note nevertheless that the modes  $S_k(\tau)$  in the Heisenberg picture are well-defined as they have the convergence factor  $e^{-2\pi\tau(2n+k)/L'}$  which eliminates high excitations. We could thus regularize the action of our local operators by introducing convergence factors in the formally divergent sums but this is not very convenient for us. In order to describe representations of the algebra generated by these local operators, we prefer to choose another regularization in which a locally defined field at a point  $w = \tau - ix$  becomes a bilocal field at points  $z$  and  $\bar{z}$  (hence formally “point splitting” by going to  $\mathbb{C} \times \mathbb{C}$ ). So, we interpret the field  $S(z, \bar{z})$  as a regularization of the local interchiral field on the cylinder where the variables  $z$  and  $\bar{z}$  might be considered independent. After such a regularization the modes of  $S(z, \bar{z})$  given in (5.42) now indeed generate states of finite energy only.

Recall now that the modes of the regularized field,  $S(z, \bar{z})$ , are elements of the Lie algebra  $\mathfrak{S}_\infty$  which is a subalgebra in the algebra  $\mathfrak{gl}'_\infty$  of infinite matrices with finite number of non-zero elements. It turns out that the modes of the local, non-regularized, field  $S_{\text{cyl.}}(x, \tau)$  belong to the algebraic completion  $\overline{\mathfrak{gl}}_\infty$  of this  $\mathfrak{gl}'_\infty$ . The Lie algebra  $\overline{\mathfrak{gl}}_\infty$  was already mentioned above in Sec. 5.1.2 and it consists of infinite matrices with finite number of diagonals having (possibly infinite number of) non-zero elements, see also App. B. Working now in the completion  $\overline{\mathfrak{gl}}_\infty$ , we can ask about the Lie algebra generated by the



local operators. We show below that all the other operators  $S_k^{(l)}$  that appear in the scaling limit of the  $H_{l'}(n)$  operators are produced from  $S_k^{(0)}$  by the conformal generators  $L_n$  and  $\bar{L}_n$ . It is of course interesting to study, more generally, commutation relations involving these operators, that is, to study the Lie algebraic structure generated from the  $S_k^{(l)}$ .

### 5.4.3 Commutation relations and the Lie algebra $\mathfrak{S}$

We first find commutation relations among the modes  $S_k^{(0)}$  with the result

$$[S_r^{(0)}, S_s^{(0)}] = J_{\alpha\beta} \sum_{n \in \mathbb{Z}} ((n-s)\bar{\psi}_n^\alpha \bar{\psi}_{r+s-n}^\beta - (n+r)\psi_n^\alpha \psi_{-(r+s)-n}^\beta). \quad (5.49)$$

Recall that the modes of the chiral stress-energy tensor can be written as  $L_k = \frac{1}{2}J_{\alpha\beta} \sum_{n \in \mathbb{Z}} \psi_n^\alpha \psi_{k-n}^\beta$ , and similarly for the anti-chiral modes. We introduce also

$$L_k^{(l)} = \frac{1}{2}J_{\alpha\beta} \sum_{n \in \mathbb{Z}} n^l \psi_n^\alpha \psi_{k-n}^\beta, \quad \bar{L}_k^{(l)} = \frac{1}{2}J_{\alpha\beta} \sum_{n \in \mathbb{Z}} n^l \bar{\psi}_n^\alpha \bar{\psi}_{k-n}^\beta, \quad l \geq 0, \quad k \in \mathbb{Z}. \quad (5.50)$$

Then, (5.49) has the form

$$[S_r^{(0)}, S_s^{(0)}] = 2\bar{L}_{r+s}^{(1)} - 2L_{-(r+s)}^{(1)} - 2s\bar{L}_{r+s}^{(0)} - 2rL_{-(r+s)}^{(0)}. \quad (5.51)$$

Note that the new modes  $L_k^{(1)}$  and  $\bar{L}_k^{(1)}$  together with  $L_n^{(0)}$  and  $\bar{L}_n^{(0)}$  generate a subalgebra:

$$2[L_n^{(1)}, L_m^{(1)}] = m(n-2m)L_{n+m}^{(1)} + m^3L_{n+m}^{(0)}, \quad (5.52)$$

$$2[\bar{L}_n^{(1)}, \bar{L}_m^{(1)}] = m(n-2m)\bar{L}_{n+m}^{(1)} + m^3\bar{L}_{n+m}^{(0)}, \quad (5.53)$$

$$[L_n^{(1)}, \bar{L}_m^{(1)}] = 0, \quad [L_n^{(1)}, \bar{L}_m^{(0)}] = 0, \quad [\bar{L}_n^{(1)}, L_m^{(0)}] = 0, \quad (5.54)$$

$$[L_n^{(1)}, L_m^{(0)}] = (n-2m)L_{n+m}^{(1)} + m^2L_{n+m}^{(0)}, \quad (5.55)$$

$$[\bar{L}_n^{(1)}, \bar{L}_m^{(0)}] = (n-2m)\bar{L}_{n+m}^{(1)} + m^2\bar{L}_{n+m}^{(0)}. \quad (5.56)$$

Note also that  $2[L_n^{(1)}, L_m^{(1)}] = m[L_n^{(1)}, L_m^{(0)}]$ .

In general, we have

$$[S_r^{(l)}, S_s^{(l')}] = \sum_{k=0}^{l+l'+1} (a_k L_{-r-s}^{(k)} + \bar{a}_k \bar{L}_{r+s}^{(k)}), \quad (5.57)$$

where  $a_k$  and  $\bar{a}_k$  are coefficients in the expansions

$$\sum_{k=0}^{l+l'+1} a_k x^k = (-1)^{l'+1} (2x+r)^l (x+r) (2x+2r+s)^{l'} \quad (5.58)$$

$$\sum_{k=0}^{l+l'+1} \bar{a}_k x^k = (-1)^{l+1} (2x-s)^{l'} (-x+s) (2x-2s-r)^l. \quad (5.59)$$

We compute next for all higher modes the action of the left and right Virasoro, *i.e.*, commutation relations between  $S_r^{(l)}$  and  $L_s^{(0)}$ ,  $\bar{L}_s^{(0)}$ :

$$\begin{aligned} 2[S_r^{(l)}, L_s^{(0)}] &= S_{r-s}^{(l+1)} - \sum_{k=0}^l (-s)^{l-k} \left[ s \binom{l+1}{k} + r \binom{l}{k} \right] S_{r-s}^{(k)}, \\ 2[S_r^{(l)}, \bar{L}_s^{(0)}] &= S_{r+s}^{(l+1)} - \sum_{k=0}^l (-s)^{l-k} \left[ s \binom{l+1}{k} - r \binom{l}{k} \right] S_{r+s}^{(k)}. \end{aligned} \quad (5.60)$$

In general, we have

$$\begin{aligned} [S_r^{(l)}, L_s^{(l')}] &= \frac{-1}{2^{l'+2}} \sum_{k=0}^{l+1} \sum_{k'=0}^{l'+1} (-s)^{l-k} \left[ s(s-r)^{l'-k'} \binom{l+1}{k} \binom{l'}{k'} + r(s-r)^{l'-k'} \binom{l}{k} \binom{l'}{k'} \right. \\ &\quad \left. + (-1)^{l'} (r+s)(-s-r)^{l'-k'} \binom{l}{k} \binom{l'+1}{k'} \right] S_{r-s}^{(k+k')} \end{aligned} \quad (5.61)$$

and a similar formula for  $[S_r^{(l)}, \bar{L}_s^{(l')}]$ .

Although it is obvious that the commutator  $[L_n^{(l)}, L_m^{(l')}]$  is expressed as a linear combination of  $L_{n+m}^{(l'')}$  with  $l'' \leq l + l'$ , its precise expression looks more complicated and we do not give it for simplicity. Finally, we conclude that the vector space with the basis  $S_n^{(l)}$ ,  $L_n^{(l)}$  and  $\bar{L}_n^{(l)}$ , with  $l \geq 0$  and  $n \in \mathbb{Z}$ , has a Lie algebra structure. We will denote *this Lie algebra as*  $\mathfrak{S}$ . An embedding of the Lie algebra  $\mathfrak{S}$  of local operators into  $\overline{\mathfrak{gl}}_\infty$  is constructed in App. B. It is noteworthy to mention that all operators that appear in repeated commutators of  $S_n^{(0)}$ , including all  $S_n^{(l)}$ , do commute with the action of  $U_q^{\text{odd}} \mathfrak{sl}(2)$ . We thus have that the action of the enveloping algebra  $U\mathfrak{S}$  commutes with  $U_q^{\text{odd}} \mathfrak{sl}(2)$ .

#### 5.4.4 The relation between the $\mathfrak{S}$ and $\mathfrak{S}_\infty$ algebras

We have so far obtained two different Lie algebras. One,  $\mathfrak{S}_\infty$ , is generated by bilinears in fermion modes, while the other,  $\mathfrak{S}$ , is generated by the modes of bilinears in fermionic fields, which are local operators. The corresponding objects such as  $S_n^{(l)}$  and so on expand on an infinite sum of bilinears in fermion modes, and we now have to face the question of the equivalence of these two descriptions. We note first that the generators of the second algebra,  $\mathfrak{S}$ , are formally infinite sums of generators of  $\mathfrak{S}_\infty$ . On the other hand, any element in the first algebra  $\mathfrak{S}_\infty$ , is expressed (by definition) as a *finite* linear combination of fermion bilinears. Therefore, in order to compare we should first take the algebraic completion  $\overline{\mathfrak{S}}_\infty$  of the Lie algebra  $\mathfrak{S}_\infty$ , that now admits infinite sums like those for  $S_n^{(l)}$ . Note that this completion belongs to  $\overline{\mathfrak{gl}}_\infty$ , see App. B. Then, we obviously have an inclusion  $\mathfrak{S} \subset \overline{\mathfrak{S}}_\infty$ .

It is interesting to go back for a while to the finite lattice and to mention that there is actually an isomorphism between the Lie algebra generated by the generalized higher Hamiltonians  $H_l(n)$ , from which we extracted operators like  $S_n^{(l)}$ , and the Lie algebra  $\mathfrak{S}_N$ . The existence of such an isomorphism is discussed in App. D.1. Recall that  $\mathfrak{S}_N$  was introduced in Sec. 4 in order to give a special basis of bilinear monoms in fermionic modes in the spin-chain representation of  $JTL_{2L}$ , and then used in Sec. 5.1 to obtain the full scaling limit of the JTL algebras in this special basis. Meanwhile, the Lie algebra generated by  $H_l(n)$ 's gives in the scaling limit the Lie algebra  $\mathfrak{S}$ . It is natural to expect that the isomorphism between the two Lie algebras for finite spin chains carries over to their scaling limit,

after taking a proper completion. Therefore, we conjecture that both Lie algebras are isomorphic:

$$\overline{\mathfrak{S}} \cong \overline{\mathfrak{S}}_\infty, \quad (5.62)$$

where we introduced the algebraic completion  $\overline{\mathfrak{S}}$  of the Lie algebra  $\mathfrak{S}$ , such that it contains infinite sums of  $S_n^{(l)}$ 's, *etc.*, and is a subalgebra in  $\overline{\mathfrak{gl}}_\infty$ . All our completions of Lie algebras are consistent with the completion  $\overline{\mathfrak{gl}}_\infty$  of  $\mathfrak{gl}'_\infty$ .

We can justify this conjecture by another, heuristic but more physical, argument. Recall that the modes  $S_k = S_k^{(0)}$  were obtained by expanding the field  $S_{\text{pl.}}(z, \bar{z})$  defined in (5.40) on the unit circle in the complex plane (or at time  $\tau = 0$  in the cylinder geometry). The higher modes  $S_k^{(l)}$  can be obtained by expansion at different moments  $\tau$ . Indeed, we see from (5.60) that the dilatation operator  $L_0 + \bar{L}_0$  generates  $S_k^{(1)}$  from  $S_k^{(0)}$ , and so on. This action can be interpreted as an expansion of  $S_{\text{pl.}}(z, \bar{z})$  on a different (non-unit) circle with the same centre. Transforming the unit-circle expansion by the conformal generators  $L_n$  and  $\bar{L}_n$  we cover an expansion of the interchiral field  $S_{\text{pl.}}(z, \bar{z})$  on the whole complex plane. We thus believe that the vector space of all higher modes  $S_n^{(l)}$ , with  $n \in \mathbb{Z}$  and  $l \geq 0$ , describes an expansion on the complex plane. On the other hand, this vector space has another basis spanned by quadratic monoms  $S_{n,m}$  in the fermions modes defined above. This suggest strongly that both Lie algebras – the one generated by  $S_n^{(l)}$  and the other,  $\mathfrak{S}_\infty$ , generated by  $S_{n,m}$  – are isomorphic.

Note finally that considering completions of the Lie algebras appearing in the scaling limit of our spin-chains is crucial if we want to describe local operators.

## 5.5 The interchiral algebra $\mathcal{S}$

We define *the interchiral algebra* for the  $\mathfrak{gl}(1|1)$  models, denoted by  $\mathcal{S}$ , as follows. Consider the completion  $\overline{\mathfrak{S}}_\infty$  of the Lie algebra  $\mathfrak{S}_\infty$  generated by the modes of the regularized interchiral field  $S(z, \bar{z})$  in its double mode expansion. The interchiral algebra  $\mathcal{S}$  is then defined as the associative algebra generated by  $\overline{\mathfrak{S}}_\infty$ , *i.e.*, it is the homomorphic image of  $U\overline{\mathfrak{S}}_\infty$ :

$$U\overline{\mathfrak{S}}_\infty \longrightarrow \mathcal{S}. \quad (5.63)$$

The defining relations in  $\mathfrak{S}_\infty$  and thus in the universal enveloping algebra  $U\overline{\mathfrak{S}}_\infty$  are computed in Sec. 5.2. The point is, there might be additional relations in the fermionic representation of the abstract algebra  $U\overline{\mathfrak{S}}_\infty$ , and the particular modules realized in  $\mathcal{H}$ , as also happens say for the Virasoro Lie algebra and its enveloping algebra  $\mathcal{V}(2)$ . This is why the map from (5.63) is not an isomorphism but a covering homomorphism of associative algebras.

We also recall the discussion in the previous subsection about relations between both Lie algebras  $\mathfrak{S}$  and  $\mathfrak{S}_\infty$ . In view of the (conjectured) isomorphism of their completions, we can give another, more physical, definition of the interchiral algebra. The interchiral algebra can be considered as the associative algebra generated by the modes of the following *local* fields: the stress-energy tensor fields  $T(x, \tau)$  and  $\bar{T}(x, \tau)$  and the interchiral field  $S_{\text{cyl.}}(x, \tau)$ , see the mode expansion in (5.44) and in (5.47), where  $x$  and  $\tau$  are the coordinates on the cylinder. The Lie algebra generated by these modes requires of course the completion,  $\overline{\mathfrak{S}}$ , in order to get indeed a definition of  $\mathcal{S}$  equivalent to the first one. Recall that the Lie algebra  $\mathfrak{S}$  was introduced at the end of Sec. 5.4.3 and the completion  $\overline{\mathfrak{S}}$  of  $\mathfrak{S}$  was introduced in Sec. 5.4.4. In the symplectic-fermion theory, the interchiral algebra is then the homomorphic image of the (representation of the) enveloping algebra  $U\overline{\mathfrak{S}}$ . The Lie algebra  $\mathfrak{S}$ , and

its completion as well, has  $S_n^{(l)}$ ,  $L_n^{(l)}$  and  $\bar{L}_n^{(l)}$ , with  $l \geq 0$  and  $n \in \mathbb{Z}$ , as its basis. The commutation relations in this basis were computed also in Sec. 5.4.3, which give relations in  $\mathcal{S}$ , but probably not all defining relations, because there should be more in the representation of  $U\mathfrak{S}$ . We write therefore

$$U\overline{\mathfrak{S}}_\infty \cong U\overline{\mathfrak{S}} \longrightarrow \mathcal{S} \quad (5.64)$$

where the arrow is again a covering homomorphism of associative algebras and  $\mathcal{S}$  is the image of  $U\overline{\mathfrak{S}}$  under this homomorphism.

Note that while the second definition of the interchiral algebra is probably better for further generalizations of the concept of interchiral algebras for other models, we will use the first, more technical (and less physically pleasant), definition of the interchiral algebra in studying its simple modules below.

We note further that for more complicated logarithmic theories like those describing scaling limit of  $\mathfrak{gl}(n|n)$  ( $n > 1$ ) periodic spin-chains, we will have a different and now faithful representation of  $JTL_N$ , and a description in terms of Lie algebras will probably not be available. Nevertheless, a description in terms of a (properly defined) interchiral algebra should still exist. This will be discussed in further work.

An important advantage in using the interchiral algebra is that it gives a convenient “vertex-operator algebra” framework<sup>9</sup> where one introduces operator-valued generating functions of formal variables  $z$  and  $\bar{z}$  or vertex operators and their OPE in order to define an algebraic structure. It is worth spending some time discussing this approach in the context of the bulk theory. We recall first the OPE of derivatives  $\psi^\alpha(z)$  and  $\bar{\psi}^\alpha(\bar{z})$  of the symplectic fermions

$$\psi^\alpha(z)\psi^\beta(w) = J^{\alpha\beta} \left( \frac{1}{(z-w)^2} - T(w) + \text{reg.} \right), \quad \bar{\psi}^\alpha(\bar{z})\bar{\psi}^\beta(\bar{w}) = J^{\alpha\beta} \left( \frac{1}{(\bar{z}-\bar{w})^2} - \bar{T}(\bar{w}) + \text{reg.} \right), \quad (5.65)$$

where we use the symplectic form  $J^{\alpha\beta}$ , with  $J^{12} = -J^{21} = 1$ , and computed the coefficient in front of the stress tensor  $T(w)$  using usual conformal-invariance arguments. The stress tensor  $T(z)$  is given in fermions as  $T(z) = \frac{1}{2}J_{\alpha\beta} : \psi^\alpha(z)\psi^\beta(z) :$  with  $J_{\alpha\beta}$  such that  $J_{\alpha\beta}J^{\beta\gamma} = \delta_\alpha^\gamma$ .

Then, OPE of two  $S(z, \bar{z})$  fields can be written using (5.65) as

$$\begin{aligned} S(z, \bar{z})S(w, \bar{w}) &= S_{\alpha\beta}\psi^\alpha(z)\bar{\psi}^\beta(\bar{z})S_{\gamma\delta}\psi^\gamma(w)\bar{\psi}^\delta(\bar{w}) \\ &= -S_{\alpha\beta}S_{\gamma\delta} \left( J^{\alpha\gamma} \left( \frac{1}{(z-w)^2} - T(w) + [\mathbf{1}]_{>2} \right) J^{\beta\delta} \left( \frac{1}{(\bar{z}-\bar{w})^2} - \bar{T}(\bar{w}) + [\bar{\mathbf{1}}]_{>2} \right) \right. \\ &\quad + J^{\alpha\gamma} \frac{1}{(z-w)^2} (: \bar{\psi}^\beta(\bar{w})\bar{\psi}^\delta(\bar{w}) : + (\bar{z}-\bar{w}) : \bar{\partial}\bar{\psi}^\beta(\bar{w})\bar{\psi}^\delta(\bar{w}) : + \dots) \\ &\quad \left. + J^{\beta\delta} \frac{1}{(\bar{z}-\bar{w})^2} (: \psi^\alpha(w)\psi^\gamma(w) : + (z-w) : \partial\psi^\alpha(w)\psi^\gamma(w) : + \dots) + \text{reg.} \right) \quad (5.66) \end{aligned}$$

where we take into account contributions of all possible single and double contractions, and denote by  $[\mathbf{1}]_{>2}$  and  $[\bar{\mathbf{1}}]_{>2}$  contributions of descendants from levels  $(n, 0)$  and  $(0, n)$  with  $n > 2$ , respectively. We

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<sup>9</sup>Our interchiral algebra strictly speaking is not a vertex-operator algebra because of the neither-chiral-nor-antichiral fields. A proper generalization of the vertex-operator algebras for our context is still required.

note that  $S_{\alpha\beta}S_{\gamma\delta}J^{\alpha\gamma}J^{\beta\delta} = -2$  and  $S_{\alpha\beta}S_{\gamma\delta}J^{\alpha\gamma} = J_{\beta\delta}$ . Therefore, the OPE is

$$S(z, \bar{z})S(w, \bar{w}) = 2 \left( \frac{1}{(z-w)^2(\bar{z}-\bar{w})^2} - \frac{2T(w)}{(\bar{z}-\bar{w})^2} - \frac{2\bar{T}(\bar{w})}{(z-w)^2} + \frac{[\mathbf{1}]_{>2}}{(\bar{z}-\bar{w})^2} + \frac{[\bar{\mathbf{1}}]_{>2}}{(z-w)^2} \right) - \frac{(z-w)\partial T(w) + \dots}{(\bar{z}-\bar{w})^2} - \frac{(\bar{z}-\bar{w})\bar{\partial}\bar{T}(\bar{w}) + \dots}{(z-w)^2} + \text{reg.} \quad (5.67)$$

We note that the right-hand side in the relation (5.51) is consistent with the OPE (5.67) in a sense that commuting two  $S(z, \bar{z})$  fields we can get only the identity and a linear combination of modes of  $T(z)$  and  $\bar{T}(\bar{z})$  and of their descendants, due to presence of the symplectic form  $J_{\alpha\beta}$  in (5.21).

The OPE of  $S(z, \bar{z})$  with  $T(w)$  and  $\bar{T}(\bar{w})$  is quite obvious because the field  $S(z, \bar{z})$  is primary for both chiral and anti-chiral stress tensors. We describe below in Sec. 6.3 the vacuum module over the interchiral algebra  $\mathcal{S}$  and some other its simple modules explicitly using this “vertex-operator” formulation.

## 5.6 The interchiral algebra in the twisted model

Of course, we could also consider the antiperiodic model (or the twisted sector) for the symplectic fermions where the modes are half-integer [24]. This corresponds to the scaling limit of the anti-periodic  $\mathfrak{gl}(1|1)$  spin-chain from Sec. 3.4. The  $JTL_N$  algebra is then replaced by  $JTL_N^{tw}$  while  $U_q^{\text{odd}}sl(2)$  is replaced by  $Usl(2)$ . The corresponding bimodule is semisimple and is given in (3.8).

We take the scaling limit of  $JTL_N^{tw}$  using Thm. 4.1.1 where it was shown that the spin-chain representation of  $JTL_N^{tw}$  is isomorphic to a representation of the enveloping algebra  $U\mathfrak{sp}_N$ . We thus need only to take the scaling limit of the Lie algebra  $\mathfrak{sp}_N$  which is quite straightforward. Repeating the analysis given in Sec. 5.1.2, we obtain in this case that  $\mathfrak{sp}_N$  has the limit generated by the same bilinears (5.14) as in the periodic case but now with  $n, m \in \mathbb{Z} - \frac{1}{2}$ . This representation of  $\mathfrak{sp}_\infty$  gives a symplectic-fermion representation of the universal enveloping algebra  $U\mathfrak{S}_\infty$  in the twisted model as well and the generators  $S_{m,n}$ ,  $T_{m,n}$ , and  $\bar{T}_{m,n}$  have similar expressions as in (5.15), with fermionic modes shifted by one-half, *i.e.*, with the indexes  $n, m$  for the generators being integers. In particular, the radical of  $\mathfrak{S}_\infty$  which is a subalgebra generated by  $S_{0,n}$ ,  $T_{m,0}$ , *etc.*, is trivially represented in this model.

Using the local operators or modes of the local fields  $S_{\text{cyl}}(x, \tau)$  and the energy-momentum tensors in the half-integer sector, we obtain the corresponding representation of the interchiral algebra in the anti-periodic model. And once again, taking then the completion  $\overline{\mathfrak{sp}}_\infty$  of  $\mathfrak{sp}_\infty$  in this half-integer sector, we obtain the representation of  $\mathcal{S}$  as the representation of  $U\overline{\mathfrak{sp}}_\infty$ , which is also a quotient of  $U\mathfrak{S}_\infty$ .

There are no zero modes in the anti-periodic model and the continuum limit of the  $Usl(2)$  generators reads as [21]

$$\tilde{Q}^a = d_{\alpha\beta}^a \sum_{n=0}^{\infty} \left( \frac{\psi_{-n-1/2}^\alpha \psi_{n+1/2}^\beta}{n+1/2} - \frac{\bar{\psi}_{-n-1/2}^\alpha \bar{\psi}_{n+1/2}^\beta}{n+1/2} \right), \quad (5.68)$$

with the matrices

$$d_{\alpha\beta}^0 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad d_{\alpha\beta}^1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad d_{\alpha\beta}^2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (5.69)$$

with  $[\tilde{Q}^a, \tilde{Q}^b] = f_c^{ab} \tilde{Q}^c$  and  $f_2^{01} = -1$ . We check then that the action of the Lie algebra  $\mathfrak{S}_\infty$  of fermionic bilinears commutes with this action of  $Usl(2)$ . Moreover, using the semisimplicity of this case we can

state the following result which is an analogue of the result in Prop. 5.3.1 about the centralizer in the integer-mode sector.

**Proposition 5.6.1.** *The centralizer  $\mathfrak{Z}_{U\mathfrak{S}_\infty}$  of the enveloping algebra  $U\mathfrak{S}_\infty$  action in the half-integer-mode (twisted) sector, which factorizes to the action of  $U\mathfrak{sp}_\infty$  in this model, is given by the representation of  $Usl(2)$  from (5.68).*

This is confirmed by the decomposition of the full partition function in the next section.

## 6 Modules over $\mathcal{S}$

In this new section, we describe the structure of simple and indecomposable modules over the interchiral algebra  $\mathcal{S}$ . We begin with describing the bimodule structure over  $U\mathfrak{S}_\infty$  and its centralizer. We then consider the Virasoro content of simple modules in Sec. 6.2 and give in Sec. 6.3 an explicit description of the vacuum module over the interchiral algebra  $\mathcal{S}$ , and the twisted vacuum module in Sec. 6.4. The generating functions of the full theories are given in Sec. 6.5. Finally, indecomposable modules over the interchiral algebra are analyzed in Sec. 6.6.

### 6.1 Bimodule in the scaling limit

We recall our result in Prop. 5.3.1 about the centralizing algebra for (the non-completed)  $U\mathfrak{S}_\infty$  – it is the same  $U_q^{\text{odd}}sl(2)$  as for any  $JTL_N$  algebra in the periodic spin-chain but its representation now admits spins of any non-negative integer value. We have thus essentially the same decomposition of  $\mathcal{H}$  as a module over  $U_q^{\text{odd}}sl(2)$  up to multiplicities in front of indecomposable direct summands. These multiplicities are now graded vector spaces and they should have a module structure over  $U\mathfrak{S}_\infty$ . In our space of scaling states where each state is a finite linear combination of basis ones, we can easily extend our analysis from [22] and obtain the bimodule structure over the pair of commuting algebras  $(U\mathfrak{S}_\infty, U_q^{\text{odd}}sl(2))$  as in Fig. 3. Note that the centralizer of  $U_q^{\text{odd}}sl(2)$  here is an associative algebra of endomorphisms of graded vector spaces  $\text{End}(\mathcal{H}) = \bigoplus_{n,m \in \mathbb{N}} \text{Hom}(\mathcal{H}_{(n)}, \mathcal{H}_{(m)})$ , where  $\mathcal{H}_{(n)}$  are finite-dimensional homogeneous subspaces of the  $n$ th energy level, and such that they commute with  $U_q^{\text{odd}}sl(2)$ . Note also that, just as in the case of finite spin-chains, we do not have the double-centralizing property. Using our definition of the centralizer in the infinite-dimensional case, the proof of the bimodule structure from Fig. 3 is rather straightforward and repeats essentially steps from [22], so we do not discuss it more here. In the figure for the bimodule in the scaling limit, each node is now a simple  $\mathfrak{S}_\infty$ -module and the towers have infinite length. As a consequence of this bimodule structure, we can study simple  $\mathfrak{S}_\infty$ -modules as scaling limits of simple  $JTL_N$ -modules. The first question is of course about the left-right Virasoro content of these scaling limits, which we discuss below.

### 6.2 The content of simple $\mathfrak{S}_\infty$ -modules

We start our analysis by discussing the left-right Virasoro  $\mathcal{V}(2) = \mathcal{V}(2) \boxtimes \overline{\mathcal{V}}(2)$  content of the simple  $JTL_N$  modules in the scaling limit. By this, we mean more precisely the representation content of the states that contribute to the scaling limit of the  $JTL_N$  modules, see the discussion about the scaling limit above in Sec. 5. It is convenient for this purpose to first evaluate the generating functions of energy levels in some of the  $JTL_N$  modules we have encountered previously. This can be done by reorganizing the fermions such that  $\psi^{1,2}$ 's correspond to even modes of new fermions and  $\bar{\psi}^{1,2}$ 's give

odd modes of the fermions, see App. B. Then, we can produce a generating function  $F_{j,(-1)^{j+1}}$  for each charged sector with  $S^z = j$  using the new fermions. Repeating an exercise similar to what can be found in [42], we obtain the character formula for  $F_{j,(-1)^{j+1}}$  which reads

$$F_{j,e^{2iK}} = \frac{1}{q^{-1/12}P(q)\bar{q}^{-1/12}P(\bar{q})} \sum_{n \in \mathbb{Z}} q^{\frac{[j+2(n+K)]^2-1}{8}} \bar{q}^{\frac{[j-2(n+K)]^2-1}{8}}. \quad (6.1)$$

We present a more general derivation of this result in App. E which is based on well-know scaling properties of the spectrum for twisted XXZ models, and will be useful in our future studies [35]. From the structure of the spin chain modules (see Sec. 3.3 and Fig. 4), we deduce then the traces  $F_{j,(-1)^{j+1}}^{(0)}$  for simple modules  $\mathcal{L}_{j,(-1)^{j+1}}$  which read, in terms of Virasoro characters, (see a calculation in App. E)

$$F_{j,(-1)^{j+1}}^{(0)} = \sum_{j_1, j_2 > 0}^* \chi_{j_1,1} \bar{\chi}_{j_2,1} \quad (6.2)$$

where the sum is done with the following constraints:

$$|j_1 - j_2| + 1 \leq j, \quad j_1 + j_2 - 1 \geq j, \quad j_1 + j_2 - j = 1 \bmod 2 \quad (6.3)$$

(note this is equivalent to treating  $j$  not as a spin but as a degeneracy, *i.e.*, setting  $j = 2s + 1$ ,  $j_i = 2s_i + 1$  and combining  $s_1$  and  $s_2$  to obtain spin  $s$ ), and we recall the characters of the simple modules with Kac labels  $(j, 1)$  over Virasoro at  $c = -2$ :

$$\chi_{j,1} = \frac{q^{(2j-1)^2/8} - q^{(2j+1)^2/8}}{\eta(q)}. \quad (6.4)$$

For instance, we have

$$\begin{aligned} F_{0,-1}^{(0)} &= 0, \\ F_{1,1}^{(0)} &= \sum_{r=1}^{\infty} \chi_{r,1} \bar{\chi}_{r,1}, \\ F_{2,-1}^{(0)} &= \sum_{r=1}^{\infty} \chi_{r,1} (\bar{\chi}_{r-1,1} + \bar{\chi}_{r+1,1}), \\ F_{3,1}^{(0)} &= \chi_{11} \bar{\chi}_{31} + \chi_{21} (\bar{\chi}_{21} + \bar{\chi}_{41}) + \sum_{r=3}^{\infty} \chi_{r,1} (\bar{\chi}_{r-2,1} + \bar{\chi}_{r,1} + \bar{\chi}_{r+2,1}), \\ &\dots \end{aligned} \quad (6.5)$$

The result (6.2) implies that, in contrast with the open case, simple modules of the  $JTL_N$  algebra do not become simple modules over  $\mathcal{V}(2)$  in the scaling limit. While in general one might expect that this scaling limit gives rise to non-fully-reducible modules over  $\mathcal{V}(2)$ , it turns out that one gets in fact a direct sum of simple modules over the full Virasoro algebra  $\mathcal{V}(2)$ . This can be checked either by explicitly working out this scaling limit as was done in [21], or by using a direct argument. We first note that the left Virasoro generators are particular endomorphisms of the module structure over the right Virasoro  $\bar{\mathcal{V}}(2)$  action. Then, taking into account possible extensions/‘glueings’ between simple  $\mathcal{V}(2)$ -modules – a module  $\mathcal{X}_{j,1}$  can be extended only by  $\mathcal{X}_{j \pm 1,1}$  into an indecomposable module, – the

left Virasoro can map vectors from subquotients  $\mathcal{X}_{j,1} \boxtimes \bar{\mathcal{X}}_{k,1}$  only to  $\mathcal{X}_{j\pm 1,1} \boxtimes \bar{\mathcal{X}}_{k,1}$ . Therefore, the character of a reducible indecomposable  $\mathcal{V}(2)$ -module has to have at least one pair of the indices  $(j, k)$  with  $j - k = 0 \pmod 2$  which is not true for the functions  $F_{n,(-1)^{n-1}}^{(0)}$  given above. We thus conclude that  $\mathcal{V}(2)$ -modules with characters in (6.2), for any fixed  $j$ , are semisimple. The foregoing analysis gives thus the left-right Virasoro content

$$\mathcal{L}_{j,(-1)^{j+1}} \mapsto \bigoplus_{j_1, j_2 > 0}^* \mathcal{X}_{j_1,1} \boxtimes \bar{\mathcal{X}}_{j_2,1}, \quad (6.6)$$

with the conditions (6.3) on the sum.

Finally, we note that having the identification (5.34) of the Hamiltonians  $H_l(0)$  with the Cartan-subalgebra elements of  $\mathfrak{sp}_{2L-2}$ , it should be possible using a Weyl character formula for fundamental representations of  $\mathfrak{sp}_{2L-2}$  to study their character asymptotics and extract the left-right Virasoro content of JTL simples. We leave this exercise for future work.

Using the bimodule structure over  $(U\mathfrak{S}_\infty, U_q^{\text{odd}}\mathfrak{sl}(2))$  described in Sec. 6.1, we conclude that the direct sums (6.6) over Virasoro simples are simple modules over  $U\mathfrak{S}_\infty$ . Below, we check this more explicitly in one example. Correspondingly, taking the completion of each direct summand in (6.6) gives a simple module over  $U\bar{\mathfrak{S}}_\infty$  and also a simple module over  $U\bar{\mathfrak{S}}$ . Such completions are therefore simple modules over the interchiral algebra  $\mathcal{S}$ , see (5.64).

### 6.3 Vacuum module over the interchiral algebra

Let  $\mathcal{X}_j$  denotes the module over  $U\mathfrak{S}_\infty$  which is obtained in the scaling limit of the JTL simple modules  $\mathcal{L}_{j,(-1)^{j+1}}$ . Suppose we wish to show directly that  $\mathcal{X}_j$  are simple modules as well over the algebra  $U\mathfrak{S}_\infty$  and that their appropriate completions are simple modules over  $\mathcal{S}$ . Our strategy consists in a few steps: (i) we state that the  $T_{m,n}$  and  $\bar{T}_{m,n}$  basis elements of  $\mathfrak{S}_\infty$  generate the algebras of endomorphisms of simple modules (as graded vector spaces) over the left and right Virasoro algebras, respectively; (ii) we then recall that the generators  $S_{m,n}$  are modes of the interchiral field  $S(z, \bar{z})$  of dimension  $(1, 1)$  and (iii) finally we use the operator-state correspondence in CFT and identify composite fields in  $S(z, \bar{z})$  and its derivatives that produce the direct summands in (6.6) for a few explicit examples.

In our analysis, we use the  $\mathfrak{sl}(2)$  action or “global  $SU(2)$  symmetry” of the symplectic fermions commonly known as Kausch’s  $\mathfrak{sl}(2)$  action [24] which we denote by  $\mathfrak{sl}(2)$ . The generators of the  $\mathfrak{sl}(2)$  in the non-chiral symplectic-fermion theory are

$$Q^a = d_{\alpha\beta}^a \left\{ i\phi_0^\alpha \psi_0^\beta + \sum_{n=1}^{\infty} \left( \frac{\psi_{-n}^\alpha \psi_n^\beta}{n} + \frac{\bar{\psi}_{-n}^\alpha \bar{\psi}_n^\beta}{n} \right) \right\} \quad (6.7)$$

where  $d_{\alpha\beta}^a$  are defined in (5.69), and with  $[Q^a, Q^b] = f_c^{ab} Q^c$  and  $f_2^{01} = -1$ . We note that the generators  $T_{m,n}$  and  $\bar{T}_{m,n}$  commute with this  $\mathfrak{sl}(2)$  action. Meanwhile, it is important to note that the generators  $S_{m,n}$  do not commute with the  $\mathfrak{sl}(2)$  and belong to the zero isospin-projection in the triplet  $\mathfrak{sl}(2)$ -module.

First, we can analyze chiral and anti-chiral sectors separately to show that the  $T_{m,n}$  and  $\bar{T}_{m,n}$  generate the algebras of endomorphisms of simple modules (as graded vector spaces) over the left and right Virasoro algebras, respectively, or that simples  $\mathcal{X}_{j,1}$  over the Virasoro Lie algebra with  $c = -2$  are also simples over the Lie algebra of  $T_{m,n}$ ’s. To analyze the chiral sector, it is technically easier



to work in a smaller space – in the chiral symplectic-fermions theory. Recall that the scaling limit of the open  $\mathfrak{gl}(1|1)$  spin-chains described in [7] and in Sec. 2.3 gives this chiral LCFT. The symmetry algebra of this theory, which is now the centralizer of the Virasoro algebra  $\mathcal{V}(2)$ , is a representation of the full quantum group  $U_q\mathfrak{sl}(2)$  at  $q = i$ . We recall that this centralizer can be equivalently described as the semi-direct product of the  $\mathfrak{gl}(1|1)$  and the enveloping algebra  $U\mathfrak{sl}(2)$ . The generators of the  $U\mathfrak{sl}(2)$  are the divided powers of the quantum-group generators and their action coincides with the action of Kausch’s global  $\mathfrak{sl}(2)$ , while the generators  $\psi_0^{1,2}$  of  $\mathfrak{gl}(1|1)$  correspond to horizontal arrows in the bimodule diagram in Fig. 2. On the other hand, the centralizer of  $U_i\mathfrak{sl}(2)$  is an associative algebra of endomorphisms of graded vector spaces  $\text{End}(\mathcal{H}) = \bigoplus_{n,m \in \mathbb{N}} \text{Hom}(\mathcal{H}_{(n)}, \mathcal{H}_{(m)})$ , where  $\mathcal{H}_{(n)}$  are finite-dimensional homogeneous subspaces of the  $n$ th energy level, and such that they commute with  $U_i\mathfrak{sl}(2)$ . This centralizer obviously contains Virasoro but it is a bigger algebra which should be constructed as a completion<sup>10</sup> of the Virasoro algebra  $\mathcal{V}(2)$ .

The chiral theory is decomposed onto modules over the product  $U\mathfrak{sl}(2) \boxtimes \mathcal{V}(2)$  of commuting algebras as in (2.11), where our space  $\mathcal{H}$  of scaling states is the direct sum  $\mathcal{H}^+ \oplus \mathcal{H}^-$  of bosonic and fermionic degrees of freedom. The full bimodule structure over the product  $U_i\mathfrak{sl}(2) \boxtimes \mathcal{V}(2)$  is given in Fig. 2. We note that the generators  $T_{m,n}$  are well-defined operators from  $\text{End}(\mathcal{H})$  just introduced and they commute with the  $U_i\mathfrak{sl}(2)$  action. They are therefore in the centralizer of  $U_i\mathfrak{sl}(2)$ . The socle of  $\mathcal{H}$  is the intersection of the kernels of the fermionic generators of the  $\mathfrak{gl}(1|1)$  part of  $U_i\mathfrak{sl}(2)$ . We can thus restrict the action of all  $T_{m,n}$ ’s onto the socle of the modules in (2.11). Using then the  $U\mathfrak{sl}(2)$  symmetry, we obtain that simple modules over the Lie algebra generated by  $T_{m,n}$ ’s are the same, as graded vector spaces, as the simple modules over the Virasoro. Indeed, the vacuum modules  $\mathcal{X}_{1,1}$  appears with multiplicity one and it is the  $U\mathfrak{sl}(2)$  invariant but all  $T_{m,n}$ ’s are also  $U\mathfrak{sl}(2)$  invariants and they thus generate the same  $\mathcal{X}_{1,1}$  from the vacuum state. Further, we take the highest-weight state of each higher multiplet for the  $U\mathfrak{sl}(2)$  and see that all  $T_{m,n}$ ’s generate from it a module identified with (isomorphic as a graded vector space to) the simple Virasoro module  $\mathcal{X}_{j,1}$ . This proves our claim that all  $T_{m,n}$ , for  $m, n \in \mathbb{Z}$ , are graded-vector spaces endomorphisms of the simple Virasoro modules  $\mathcal{X}_{j,1}$ , for  $j \geq 1$ . Moreover, we also check that the fermion bilinears  $T_{m,n}$  are the only bilinears that commute with  $U_i\mathfrak{sl}(2)$ . They are therefore generators of the centralizer of  $U_i\mathfrak{sl}(2)$ . This means that simple modules over Virasoro are also simple modules over the Lie algebra with the basis given by  $T_{m,n}$ . We use the same arguments to show that the Lie algebra with the basis given by  $\bar{T}_{m,n}$  has simple modules  $\bar{\mathcal{X}}_{j,1}$ .

In the full non-chiral theory, the left (or right) Virasoro algebra has the same isomorphism classes of simple and indecomposable modules as in the chiral theory, see a discussion in Sec. 6.6.1 below. We thus have essentially the same representation of  $T_{m,n}$  – the same expression for the generators as in the chiral theory but acting in a much bigger space, so a difference is only in ‘multiplicities’ which are infinite now and correspond to modules over the right Virasoro  $\bar{\mathcal{V}}(2)$ . Summarizing, we give the following statement. The simple modules  $\mathcal{X}_{j,1} \boxtimes \bar{\mathcal{X}}_{k,1}$  over the left-right Virasoro algebra  $\mathcal{V}(2)$  are also simple modules over the Lie algebra with the basis given by  $T_{m,n}$  and  $\bar{T}_{m,n}$ , with  $m, n \in \mathbb{Z}$ .

We then continue by analyzing the vacuum module over  $\mathcal{S}$  which contains the identity field. By

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<sup>10</sup>The completion of the Virasoro contains in particular projectors onto a fixed,  $k$ th, energy subspace. Such operators are obviously in the centralizer of  $U_i\mathfrak{sl}(2)$  but they are not in the universal enveloping of the Virasoro Lie algebra. Of course, there is no double-centralizing property in infinite-dimensional spaces and this is why an appropriate completion is necessary for describing centralizers of quantum groups in the CFTs, but how to define the appropriate completion precisely is a subject of a further work.

definition, the vacuum module is a vector space generated from one state – the vacuum  $\Omega$  – by those  $S_{n,m}$ ,  $T_{n,m}$  and  $\bar{T}_{n,m}$  that have negative indexes  $n$  and  $m$  (note that we used here our first definition of the interchiral algebra from Sec. 5.5). It turns out that this vector space as a  $\mathfrak{S}_\infty$ -module is decomposed over the left-right Virasoro exactly like  $\mathcal{X}_1$ . We stated just above that the  $T_{n,m}$  and  $\bar{T}_{n,m}$  generators of  $\mathfrak{S}_\infty$  generate the endomorphism algebras of a simple Virasoro module  $\mathcal{X}_{j,1}$  or  $\bar{\mathcal{X}}_{j,1}$ , respectively. Therefore, to extract the Virasoro content of the vacuum module and compare it with  $\mathcal{X}_1$  we only need to identify highest-weight vectors for  $\mathcal{V}(2)$  which are generated from the vacuum state by the  $S_{n,m}$  basis elements. Equivalently, using the operator-state correspondence, we are going to identify left-right Virasoro primary fields generated from the identity by a field having these  $S_{n,m}$  as its modes. Such a field was identified in Sec. 5.4.2 with the interchiral field  $S(z, \bar{z}) = S_{\alpha\beta} \psi^\alpha(z) \bar{\psi}^\beta(\bar{z})$ . The  $S(z, \bar{z})$  is a primary field and corresponds to the state

$$|S\rangle = \lim_{z, \bar{z} \rightarrow 0} S(z, \bar{z}) \Omega$$

which belongs to the subspace  $\mathcal{X}_{2,1} \boxtimes \bar{\mathcal{X}}_{2,1}$ . Note that this subspace is in the triplet sector with respect to  $\mathfrak{sl}(2)$ . Then,  $T_{n,m}$  and  $\bar{T}_{n,m}$  generate this  $\mathcal{V}(2)$ -module from the highest-weight state  $|S\rangle$ . We thus obtained first two terms in the decomposition onto left-right Virasoro modules

$$\mathcal{X}_1|_{\mathcal{V}(2)} = \bigoplus_{j \geq 1} \mathcal{X}_{j,1} \boxtimes \bar{\mathcal{X}}_{j,1}, \quad (6.8)$$

which is the direct sum in (6.6) for  $j = 1$ . Highest-weight vectors in other direct summands in (6.8) can be constructed by taking appropriate composite fields in  $S(z, \bar{z})$  and in its derivatives  $\partial^j \bar{\partial}^j S(z, \bar{z})$  applied to the vacuum  $\Omega$  in the limit  $z, \bar{z} \rightarrow 0$ . So, the next primary field from the decomposition (6.8) should have conformal dimensions  $(3, 3)$  and it might be identified with the composite field  $:\partial \bar{\partial} S(z, \bar{z}) S(z, \bar{z}):$ , up to descendants of  $S(z, \bar{z})$  and of the identity field on the level  $(3, 3)$ . We note that this highest-weight state belongs to a 5-dimensional  $\mathfrak{sl}(2)$ -module. This result can be obtained by a direct calculation using the double mode expansion (5.42) together with normal-ordering prescriptions or using OPE formulas given in (5.67).

The analysis can be continued to construct in a similar way states belonging to higher  $\mathfrak{sl}(2)$ -multiplets and contributing thus to new primary fields. Using (6.7), we obtain that the composite field

$$:\prod_{j=0}^n \partial^j \bar{\partial}^j S(z, \bar{z}): \quad (6.9)$$

has the conformal weight  $(\Delta_{n+2,1}, \Delta_{n+2,1})$  and it is in a  $(n+1)$ -dimensional  $\mathfrak{sl}(2)$ -module, up to contributions from lower  $\mathfrak{sl}(2)$ -multiplets<sup>11</sup>. This state therefore belongs to the direct sum  $\oplus_{j=1}^{n+2} \mathcal{X}_{j,1} \boxtimes \bar{\mathcal{X}}_{j,1}$ . By induction, this finally gives the decomposition (6.8).

We next observe that the state

$$|S^2\rangle = \lim_{z, \bar{z} \rightarrow 0} :S^2(z, \bar{z}): \Omega$$

is on the level  $(2, 2)$  of the direct sum  $\mathcal{X}_{2,1} \boxtimes \bar{\mathcal{X}}_{2,1} \oplus \mathcal{X}_{1,1} \boxtimes \bar{\mathcal{X}}_{1,1}$ . This means that the square of the interchiral field maps back to the identity module over  $\mathcal{V}(2)$ . A similar analysis for states (6.9) living

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<sup>11</sup>Strictly speaking, the composite field constructed belongs actually to a direct sum  $\oplus_{j=0}^{n/2} C_j$  where  $C_j$  denotes the  $(2j+1)$ -dimensional  $\mathfrak{sl}(2)$ -module.

in higher  $\mathfrak{sl}(2)$  multiplets shows that our vector space generated by composite fields in  $S(z, \bar{z})$  and its derivatives is indeed a simple module.

We conclude that the direct sum (6.6) for  $j = 1$  is a simple module over  $U\mathfrak{S}_\infty$ . Recalling our discussion about a (conjectured) isomorphism of two different Lie algebras,  $\overline{\mathfrak{S}}_\infty$  and  $\overline{\mathfrak{S}}$ , given in Sec. 5.4.4, we can also conclude that a completion of each direct summand in the decomposition (6.8) of  $\mathcal{X}_1$  over left-right Virasoro is the vacuum module over the interchiral algebra  $\mathcal{S}$  and it is a simple module. We choose here a completion that contains all images under the action of  $\overline{\mathfrak{gl}}_\infty$  which was introduced in App. B, where we also give an embedding of  $\overline{\mathfrak{S}}_\infty$  into the  $\overline{\mathfrak{gl}}_\infty$ . The completion of modules with respect to  $\overline{\mathfrak{gl}}_\infty$  can be also described as a completion in a formal topology, where all subspaces  $\bigoplus_{\mathbb{N} \times \mathbb{N}/M} \mathcal{H}^{n,m} \subset \mathcal{H}$ , for a finite subset  $M \subset \mathbb{N} \times \mathbb{N}$ , are declared to be the fundamental system of neighbourhoods of zero. Then, a completion of  $\mathcal{H}$  in this formal topology is the direct product  $\prod_{\mathbb{N} \times \mathbb{N}} \mathcal{H}^{n,m}$  of its homogeneous, or fixed energy-spin, subspaces  $\mathcal{H}^{n,m}$  (see for example [42].) Note that we do not need to consider a completion of the full direct sum because different terms in this direct sum belong to different multiplets for  $\mathfrak{sl}(2)$ , *i.e.*, they have different isospins, and all the generators of  $\overline{\mathfrak{gl}}_\infty$  and therefore the local interchiral field as well generate only states of a fixed finite isospin, in contrast to the eigenvalue of  $L_0 + \bar{L}_0$  which is not fixed under the action of the local operators  $S_n^{(l)}$ . We finally give the conjecture that taking appropriate completions (those in the formal topology – such completions then contain images of  $\overline{\mathfrak{gl}}_\infty$  generators and therefore images of the local operators  $S_n^{(l)}$ ) of the direct summands in (6.6) for all finite  $j$  gives simple modules over  $\mathcal{S}$ .

## 6.4 The twisted vacuum module

We finally discuss an example of a simple module over the interchiral algebra  $\mathcal{S}$  in the sector with half-integer fermionic modes. Recall that we have found in Sec. 5.6 an action of the interchiral algebra  $\mathcal{S}$  in the twisted model of symplectic fermions. The full symmetry algebra in this case is the  $U\mathfrak{sl}(2)$  as it was stated in Prop. 5.6.1. We define then the twisted vacuum module over the interchiral algebra as the space of  $\mathfrak{sl}(2)$  invariants. The structure of the twisted vacuum module can be described in a way parallel to Sec. 6.2 and 6.3. The generating function of levels obtained as in [26] now involves the characters

$$\chi_{j,2} = \frac{q^{(j-1)^2/2} - q^{(j+1)^2/2}}{\eta(q)} \quad (6.10)$$

and we have the identity for the Kac character, see App. E,

$$K_{r,2j+2} = \sum_{s=0}^j \chi_{r-j+2s,2}. \quad (6.11)$$

This time, the simple modules of  $JTL_N^{tw}$  are obtained with a single subtraction according to (3.9), so their left-right Virasoro  $\mathfrak{V}(2)$  content is

$$F_{j,(-1)^j}^{(0)} = \mathcal{F}_{j,(-1)^j} - \mathcal{F}_{j+2,(-1)^j} = \sum_{r=1}^{\infty} \chi_{r,2} (\bar{\chi}_{r-j,2} + \bar{\chi}_{r-j+2,2} + \dots + \bar{\chi}_{r+j,2}). \quad (6.12)$$

The  $JTL_N^{tw}$  modules are all semi-simple in this model, and so are the  $\mathfrak{V}(2)$  modules, according to the decomposition (3.8). We thus obtain the left-right Virasoro content in the scaling limit

$$\mathcal{L}_{j,(-1)^j} \mapsto \bigoplus_{r \geq 1} \mathcal{X}_{r,2} \boxtimes (\bar{\mathcal{X}}_{r-j,2} \oplus \bar{\mathcal{X}}_{r-j+2,2} \oplus \dots \oplus \bar{\mathcal{X}}_{r+j,2}), \quad (6.13)$$

which is again a direct sum of infinite number of simple modules over  $\mathbf{V}(2)$ .

Let  $\mathcal{X}_j^{tw}$  denotes the module over  $U\mathfrak{S}_\infty$  (or  $U\mathfrak{sp}_\infty$ ) which is obtained in the scaling limit of the  $JTL^{tw}$  simple modules  $\mathcal{L}_{j,(-1)^j}$ . With an analysis similar to one in Sec. 6.3, we generate the vacuum module applying by composite fields in  $S(z, \bar{z})$  and its derivatives on the vacuum state, the highest-weight state in  $\mathcal{X}_{1,2} \boxtimes \bar{\mathcal{X}}_{1,2}$ . The vacuum module  $\mathcal{X}_0^{tw}$  decomposition onto left-right Virasoro modules is

$$\mathcal{X}_0^{tw}|_{\mathbf{V}(2)} = \bigoplus_{j \geq 1} \mathcal{X}_{j,2} \boxtimes \bar{\mathcal{X}}_{j,2} \quad (6.14)$$

and it is an irreducible module over  $U\mathfrak{sp}_\infty$ . We do not give details of calculations as their essentially repeat the previous. Correspondingly, an appropriate completion of this module, see a discussion at the end of Sec. 6.3, describes the vacuum module over  $\mathcal{S}$ .

## 6.5 The full generating functions

Mostly for pedagogical reasons, we discuss here briefly generating functions of levels for the symplectic fermion theory, interpreted in terms of our algebraical structures.

The generating function of levels for the periodic  $\mathfrak{gl}(1|1)$  model – that is, the left hand side of (E1) reads, from the analysis in [26],

$$Z = F_{0,-1} + 2 \sum_{j \text{ even}} F_{j,-1} + 2 \sum_{j \text{ odd}} F_{j,1} \quad (6.15)$$

(in the reinterpretation as the partition function of the XX model with appropriate twists, the factors 2 arise from the  $\mathbb{Z}_2$  spin-flip symmetry, so the summation index  $j$  is positive). Elementary algebra using

$$F_{2n,-1}^{(0)} + F_{2n+1,1}^{(0)} = F_{2n,-1} - F_{2n+1,1} \quad (6.16)$$

leads to

$$Z = 4 \sum_{j=0}^{\infty} j F_{j,(-1)^{j+1}}^{(0)}. \quad (6.17)$$

This formula is a direct translation of the bimodule structure discussed in Sec. 3.3: the same decomposition holds in fact for a finite system, with  $F^{(0)}$  the trace over the simple modules over  $JTL_N$ . Note that it is similar to the formula obtained in [7] for the boundary theory

$$Z = 4 \sum_{j=0}^{\infty} j \chi_{j,1}.$$

In this case, the degeneracy  $4j$  arose as dimensions of irreducible representations of the full quantum group, which coincides with the algebra  $\mathcal{A}_{1|1}$ . In the periodic case, we lose the full quantum group symmetry and retain only  $U_q^{\text{odd}} \mathfrak{sl}(2)$  on the lattice. This leads, nevertheless, to the same  $4j$  for the simple modules:  $\mathcal{L}_{j,(-1)^{j+1}}$  in the periodic case,  $\mathcal{X}_{j,1}$  in the open case.

Finally, note that

$$Z = 4 \sum_{j=0}^{\infty} j F_{j,(-1)^{j+1}}^{(0)} = 4 \left| \sum_{j=0}^{\infty} j \chi_{j,1} \right|^2 = 4 \left| q^{1/12} \prod_{n=1}^{\infty} (1 + q^n)^2 \right|^2. \quad (6.18)$$

The right hand side is now seen to coincide with the generating function for the level of symplectic fermions with periodic boundary conditions indeed [24]. Of course, the corresponding partition function of doubly periodic symplectic fermions on the torus vanishes exactly due to the  $\mathfrak{gl}(1|1)$  symmetry; it is thus trivially modular invariant. The generating function  $Z$  in (6.15) is not modular invariant, nor does it have to be.

In the anti-periodic case or twisted model, the generating function of levels is now

$$Z = \mathcal{F}_{0,1} + 2 \sum_{\text{odd } j>0} \mathcal{F}_{j,-1} + 2 \sum_{\text{even } j>0} \mathcal{F}_{j,1} \quad (6.19)$$

corresponding to

$$Z = \sum_{j=0}^{\infty} (j+1) F_{j,(-1)^j}^{(0)}, \quad (6.20)$$

in accordance with the decomposition (3.8). The  $(j+1)'s$  are now dimensions of the  $U\mathfrak{sl}(2)$ -modules; the  $\mathfrak{gl}(1|1)$  symmetry is broken. As before, we can rewrite

$$Z = \left| \sum_{j=1} j \chi_{j,2} \right|^2 = \left| q^{-1/24} \prod_{n=1}^{\infty} (1 + q^{n+1/2})^2 \right|^2 \quad (6.21)$$

which is a well known expression in this sector as well.

## 6.6 Indecomposable modules over $\mathcal{S}$

To proceed, we must now discuss the scaling limit of the indecomposable  $JTL_N$ -modules  $\widehat{\mathcal{P}}_j$ , which are indecomposable  $U\mathfrak{S}_{\infty}$ -modules denoted by  $\mathcal{Y}_j$ , and compare it with potentially similar structure in the symplectic fermion theory. This will be facilitated by a preliminary discussion of the latter.

### 6.6.1 The structure of the Virasoro representations in symplectic fermions

We already reminded the reader of the known results about the chiral conformal field theory of symplectic fermions in Sec. 2.3. We now turn to a similar analysis of the non-chiral theory and its decomposition over the left-right Virasoro  $\mathcal{V}(2)$ . In this section, we will denote our space of scaling states in the bulk theory by  $\mathcal{H}$  in order to not be confused with the decomposition (2.11) of the chiral theory.

In the integer-mode sector, the space of states  $\mathcal{H}$  for the non chiral theory is decomposed into a bosonic sector  $\mathcal{H}^+$  and the fermionic one  $\mathcal{H}^-$  with the  $\mathfrak{sl}(2)$  and left-right Virasoro  $\mathcal{V}(2)$  content [43]

$$\mathcal{H}^+|_{\mathfrak{sl}(2) \boxtimes \mathcal{V}(2)} = \bigoplus_{k \in \mathbb{N}_0} C_k \boxtimes \mathcal{H}_k^+, \quad \mathcal{H}^-|_{\mathfrak{sl}(2) \boxtimes \mathcal{V}(2)} = \bigoplus_{k \in \mathbb{N} - \frac{1}{2}} C_k \boxtimes \mathcal{H}_k^-, \quad (6.22)$$

where  $C_n$  denotes now a  $(2n+1)$ -dimensional  $\mathfrak{sl}(2)$ -module of the isospin  $n$ , and  $\mathcal{H}_k^{\pm}$  are  $\mathcal{V}(2)$ -modules which we describe below.

With the left and right  $\mathcal{V}(2)$ -modules  $\mathcal{H}_k^{\pm, (l)}$  and  $\mathcal{H}_k^{\pm, (r)}$  denoting restriction of  $\mathcal{H}_k^{\pm}$  on  $\mathcal{V}(2)$  and  $\overline{\mathcal{V}}(2)$ , respectively, the bosonic components  $\mathcal{H}_k^+$  have the decompositions, with non-negative integer  $k$ ,

$$\mathcal{H}_k^{+, (l)} = \bigoplus_{a=-k}^k \bigoplus_{n>k-a} \mathcal{P}_{n,1} \boxtimes \bar{\mathcal{X}}_{n+2a,1}, \quad \mathcal{H}_k^{+, (r)} = \bigoplus_{a=-k}^k \bigoplus_{n>k-a} \mathcal{X}_{n,1} \boxtimes \bar{\mathcal{P}}_{n+2a,1}, \quad k \in \mathbb{N}_0, \quad (6.23)$$

and the fermionic components  $\mathcal{H}_k^-$  decompose as, with a positive half-integer  $k$ ,

$$\mathcal{H}_k^{-,(l)} = \bigoplus_{a=-k}^k \bigoplus_{n>k-a} \mathcal{P}_{n,1} \boxtimes \bar{\mathcal{X}}_{n+2a,1}, \quad \mathcal{H}_k^{-,(r)} = \bigoplus_{a=-k}^k \bigoplus_{n>k-a} \mathcal{X}_{n,1} \boxtimes \bar{\mathcal{P}}_{n+2a,1}, \quad k \in \mathbb{N} - \frac{1}{2}, \quad (6.24)$$

where we use the exterior tensor product  $\boxtimes$  for two Virasoro modules in order to show that ‘multiplicities’ of left- or right-Virasoro staggered modules in the decompositions are simple subquotients over right or left Virasoro, respectively. We stress that direct summands in (6.23) and (6.24) are not  $\mathcal{V}(2)$ -modules and the rest of this section is devoted to describing a subquotient structure for the product of the two Virasoro algebras.

The sector with the trivial  $\mathfrak{sl}(2)$  action ( $k = 0$ ) has a particular interest because it contains the vacuum state  $\Omega$  and its logarithmic partner  $\omega$  ( $L_0\omega = \bar{L}_0\omega = \Omega$ ),

$$\mathcal{H}_0^{+,(l)} = \bigoplus_{n \geq 1} \mathcal{P}_{n,1} \boxtimes \bar{\mathcal{X}}_{n,1}, \quad \mathcal{H}_0^{+,(r)} = \bigoplus_{n \geq 1} \mathcal{X}_{n,1} \boxtimes \bar{\mathcal{P}}_{n,1}, \quad \Omega, \omega \in \mathcal{P}_{1,1} \boxtimes \bar{\mathcal{X}}_{1,1}. \quad (6.25)$$

These two states are the only bosonic states with conformal dimension  $(0,0)$ . There are two other states of the same conformal dimension in the fermionic sector  $\mathcal{H}_{1/2}^-$  in agreement with [44].

We use the decompositions (6.23) and (6.24) over the left and right Virasoro in studying the subquotient structure over their product  $\mathcal{V}(2)$ . We first note an obvious fact that right-Virasoro algebra elements are represented on each  $\mathcal{H}_k^\pm$  as intertwining operators for the left-Virasoro action, *i.e.*, there is a homomorphism from the universal enveloping algebra  $U(\bar{\mathcal{V}}(2))$  to  $\text{End}_{\mathcal{V}(2)}(\mathcal{H}_k^\pm)$ . Therefore, we should begin with a description of the space  $\text{Hom}_{\mathcal{V}(2)}(\mathcal{P}_{n,1}, \mathcal{P}_{n',1})$ . From the subquotient structure (2.12) of the staggered modules, we deduce that the Hom-space is one-dimensional only for  $n = n'$  or  $n = n' \pm 1$  and zero-dimensional otherwise. In the case  $n = n'$ , the image of a basis element in the Hom-space is the simple submodule  $\mathcal{X}_{n,1}$  while the image in the case  $n = n' + 1$  is the indecomposable Kac module  $\mathcal{X}_{n,1} \rightarrow \mathcal{X}_{n+1,1}$  and the image in the case  $n = n' - 1$  is the contragredient Kac module  $\mathcal{X}_{n,1} \rightarrow \mathcal{X}_{n-1,1}$ . Analysing then all possible endomorphisms on  $\mathcal{H}_k^{\pm,(l)}$  respecting the left Virasoro and combining with the decomposition of  $\mathcal{H}_k^{\pm,(r)}$  over the right Virasoro, we end up in diagrams depicting a subquotient structure over the product  $\mathcal{V}(2)$  of two Virasoro algebras for each  $\mathcal{H}_k^\pm$ .

The simplest is for  $k = 0$ , where we obtain the structure represented on the left in Fig. 5 using the decompositions (6.25). On the right in Fig. 5, we show the subquotient structure for  $k = 1/2$  as well, which is in the fermionic sector  $\mathcal{H}^-$ . For  $k = 1$  meanwhile we have the structure represented on Fig. 6. More detailed analysis of the  $\mathcal{V}(2)$ -module structure on each  $\mathcal{H}_k^\pm$  can be also found in a companion paper [43].

Each node with a coordinate  $(\bar{n}, n')$  in the diagrams on Fig. 5 and Fig. 6 corresponds to a simple subquotient over  $\mathcal{V}(2) = \mathcal{V}(2) \boxtimes \bar{\mathcal{V}}(2)$  with the lowest conformal weight  $(\Delta_{n',1}, \bar{\Delta}_{n,1})$  and arrows show the action of both Virasoro algebras – the Virasoro  $\mathcal{V}(2)$  acts in the vertical direction (preserving the coordinate  $\bar{n}$ ), while the right Virasoro  $\bar{\mathcal{V}}(2)$  acts in the horizontal way. Some values  $(\bar{n}, n')$  occur twice and those nodes/subquotients are separated slightly for clarity; we denote top subquotients by  $\diamond$ , bottom ones by  $\blacklozenge$ , and subquotients in the middle level that have incoming as well as outgoing arrows are denoted by  $\bullet$ , in order to make reading the diagrams easier. We also note that some horizontal arrows (of the same direction) connecting a  $\diamond$  with a pair of  $\bullet$ ’s at the same coordinate or a pair of  $\bullet$ ’s with a  $\blacklozenge$ , say connecting  $(\bar{1}, 3)$  with two nodes at  $(\bar{2}, 3)$  in Fig. 6, are actually doubled which means that right-Virasoro elements corresponding to the pair of arrows with the same source/sink

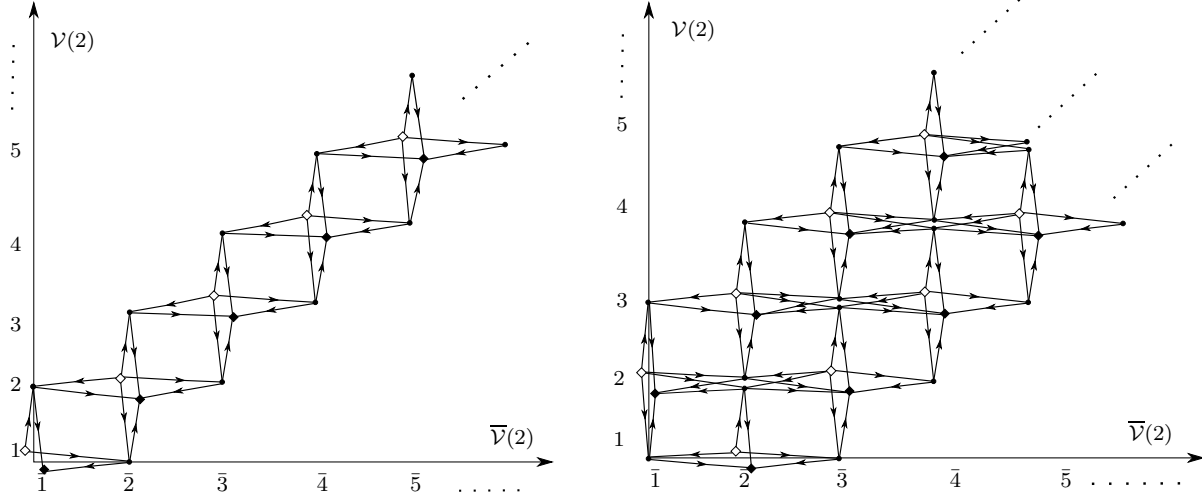


Figure 5: Module structure over  $\mathcal{V}(2) = \mathcal{V}(2) \boxtimes \overline{\mathcal{V}}(2)$  for the vacuum sector  $\mathcal{H}_0^+$  (with zero  $\mathfrak{sl}(2)$ -isospin) on the left diagram while the right one is for the doublet-sector  $\mathcal{H}_{1/2}^-$ . Each node with a coordinate  $(\bar{n}, n')$  is a simple subquotient over  $\mathcal{V}(2)$  with the conformal weight  $(\Delta_{n',1}, \bar{\Delta}_{n,1})$ . Vertical arrows represent the action of the left Virasoro  $\mathcal{V}(2)$  and horizontal arrows of the right Virasoro  $\overline{\mathcal{V}}(2)$ .

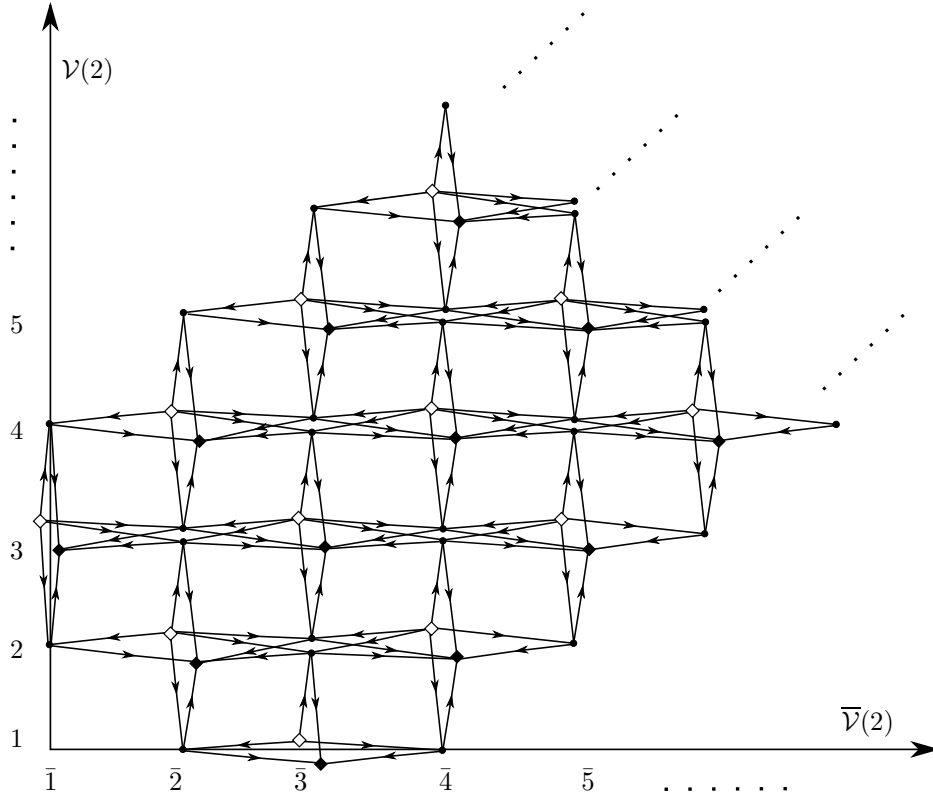


Figure 6:  $\mathcal{V}(2)$ -module subquotient structure for the triplet-sector  $\mathcal{H}_1^+$ . Each node with a coordinate  $(\bar{n}, n')$  is a simple subquotient over  $\mathcal{V}(2) = \mathcal{V}(2) \boxtimes \overline{\mathcal{V}}(2)$  with the conformal weight  $(\Delta_{n',1}, \bar{\Delta}_{n,1})$ . Vertical arrows represent the action of the left Virasoro  $\mathcal{V}(2)$  and horizontal arrows of the right Virasoro  $\overline{\mathcal{V}}(2)$ .

map to/from a fixed linear combination of the pair of subquotients depicted by  $\bullet$ . We use such a linear combination for each pair of  $\bullet$ 's that corresponds to a basis adapted for the left-*Virasoro* action<sup>12</sup> that has no (vertical) doubled arrows and is explicitly decomposed onto diagrams from (2.12). We finally assume in our diagrams that right-*Virasoro* elements map states from two neighbour  $\diamond$ 's, say  $(\bar{1}, 3)$  and  $(\bar{3}, 3)$  in Fig. 6, to linearly independent subquotients of the same coordinate, say  $(\bar{2}, 3)$ , in the middle level.

The indecomposable staggered  $\mathcal{V}(2)$ -modules  $\mathcal{P}_{n',1}$  introduced in (2.12) and appearing in the decompositions (6.23) and (6.24) can be recovered by ignoring all the horizontal arrows, while staggered  $\bar{\mathcal{V}}(2)$ -modules  $\bar{\mathcal{P}}_{n,1}$  are obtained by ignoring all the vertical arrows in the diagrams and taking an appropriate linear combination of two simple subquotients for each pair of middle-level nodes denoted by  $\bullet$  and sharing the same coordinate  $(\bar{n}, n')$ .

We note that the interchiral field  $S(z, \bar{z})$  from (5.40) generating our interchiral algebra  $\mathcal{S}$  belongs to the submodule identified in the diagram for the triplet sector in Fig. 6 with the node  $\blacklozenge$  at the position  $(\bar{2}, 2)$ .

We will comment on the similarity and consistency of these figures with our results of the (limits of)  $JTL_N$ -modules in the next subsection. Notice that the way the left and right indecomposables are glued together in (the vacuum sector of) the non-chiral theory is similar that what is observed for super WZW models on  $\mathfrak{gl}(1|1)$  and  $\mathfrak{su}(2|1)$  [20].

Finally, we stress that the symplectic fermion theory also admits action of  $\mathfrak{gl}(1|1)$  which connects in particular the bosonic  $\mathcal{H}^+$  and fermionic sectors  $\mathcal{H}^-$  of the theory. Its action is quite straightforward, and we refrain from discussing it for simplicity.

## 6.6.2 Indecomposable $U\mathfrak{S}_\infty$ and $\mathcal{V}(2)$ -modules

The main objective of this section is to show that the structure of  $\mathcal{V}(2)$ -modules described above is consistent with the scaling limit of the bimodule over  $JTL_N$  and  $\mathfrak{Z}_{JTL}$ . We recall that in Prop. 5.3.1 and in Sec. 6.1 we stated that the centralizer of  $U\mathfrak{S}_\infty$  is essentially the same as for  $JTL_N$  algebras – it is given by the “integer-spin” image of  $U_q^{\text{odd}}\mathfrak{sl}(2)$ . The structure of indecomposables for  $U\mathfrak{S}_\infty$  should be thus also essentially the same with a difference that diagrams have now infinitely many components. We present corresponding diagrams in Fig. 7. Comparing the structure of modules over  $\mathcal{V}(2)$  and  $U\mathfrak{S}_\infty$ , we then make conclusions on the structure of indecomposable modules over the interchiral algebra  $\mathcal{S}$ .

Recall that we denote the scaling limits of indecomposable spin-chain modules  $\widehat{\mathcal{P}}_j$  by  $\mathcal{Y}_j$ . These are modules over  $U\mathfrak{S}_\infty$ . Having obtained in Sec. 6.2 the content under  $\mathcal{V}(2)$  of the simple subquotients, we next study a filtration<sup>13</sup> of  $\mathcal{Y}_j$  by  $\mathcal{V}(2)$ -modules which were described in Sec. 6.6.1. We expect that the  $JTL$ -modules  $\widehat{\mathcal{P}}_j$  on each sector  $\mathcal{H}_{[j]}$  become, in the scaling limit, indecomposable modules under  $\mathcal{V}(2)$  with the same subquotient structure, leading to Fig. 7. Of course, a crucial point is that, in this figure, a node stands now for an infinite sum of products of representations of the two Virasoro algebras. This agrees with the final results of Sec. 6.2 and Sec. 6.6.1. Indeed, the  $\mathcal{V}(2)$ -modules (6.22)

<sup>12</sup>Of course, it is a matter of a convention, and we could choose a basis where horizontal arrows are not doubled but the vertical ones would be doubled. It is important to note that there exists no a basis without the doubled arrows in the corresponding diagram.

<sup>13</sup>Recall that a filtration of  $A$ -module  $M$  by its submodules  $M_i$ , with  $0 \leq i \leq n$ , is called a sequence of embeddings  $0 = M_0 \subset M_1 \subset \cdots \subset M_i \subset \cdots \subset M_{n-1} \subset M_n = M$ .



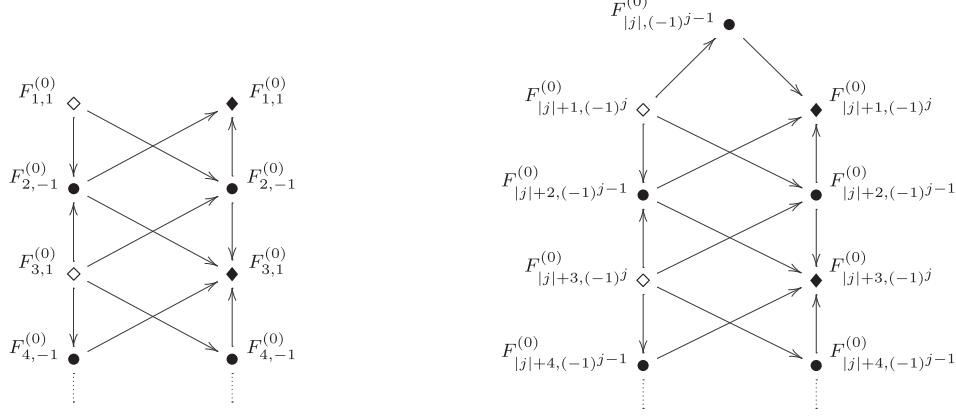


Figure 7: Scaling limit of indecomposable spin-chain modules  $\widehat{\mathcal{P}}_j$  for  $j = 0$  on the left and  $j \neq 0$  on the right side. These are indecomposable  $U\mathfrak{S}_\infty$ -modules  $\mathcal{Y}_j$  with simple subquotients.

$\mathcal{H}|_{\mathfrak{h}=j} \equiv \mathcal{H}_{(j)} = \bigoplus_{k \geq |j|} \mathcal{H}_k^+$  in each sector  $\mathfrak{h} = j$  have a filtration

$$\text{Soc}(\mathcal{H}_{(j)}) \subset \text{Mid}(\mathcal{H}_{(j)}) \subset \mathcal{H}_{(j)} \rightarrow \text{Top}(\mathcal{H}_{(j)}) \rightarrow 0 \quad (6.26)$$

by  $\mathcal{V}(2)$ -submodules (which are described below) consistent with the structure of  $\mathcal{Y}_j$  presented on Fig. 7, where each node is described by formulas in (6.2) and (6.5), and the diagram is an infinite analogue of the finite ladders in Fig. 4. In particular, we have the left Virasoro structure on  $\mathcal{H}$  at  $j = 0$ , forgetting about the  $\mathfrak{sl}(2)$  content for a while,

$$\mathcal{H}_{(0)}^{(l)} = \bigoplus_{n,m \geq 0} (2 \min(n, m) + 1) \mathcal{P}_{2n+1,1} \boxtimes \bar{\mathcal{X}}_{2m+1,1} \oplus \bigoplus_{n,m \geq 1} 2 \min(n, m) \mathcal{P}_{2n,1} \boxtimes \bar{\mathcal{X}}_{2m,1}, \quad (6.27)$$

and the right structure  $\mathcal{H}^{(r)}$  at the grade  $\mathfrak{h} = 0$  is given by the substitutions  $\mathcal{P} \rightarrow \mathcal{X}$  and  $\bar{\mathcal{X}} \rightarrow \bar{\mathcal{P}}$ . This decomposition allows us immediately to compare particular submodules/quotients – the terms in the filtration (6.26) – with the ones in  $\mathcal{Y}_0$  on Fig. 7. First, the socle (the maximal semisimple submodule)  $\text{Soc}(\mathcal{Y}_0)$  has the Virasoro character

$$\text{char} [\text{Soc}(\mathcal{Y}_0)] = \sum_{j \text{ odd}} F_{j,1}^{(0)} = \sum_{j_1, j_2 \geq 0} (2 \min(j_1, j_2) + 1) \chi_{2j_1+1,1} \bar{\chi}_{2j_2+1,1} + \sum_{j_1, j_2 \geq 1} 2 \min(j_1, j_2) \chi_{2j_1,1} \bar{\chi}_{2j_2,1}$$

which coincides with the character of  $\text{Soc}(\mathcal{H}_{(0)})$  easily extracted from (6.27). The same is true for the top parts (the maximal semisimple quotient)  $\text{Top}$  of  $\mathcal{H}_{(0)}$  and  $\mathcal{Y}_0$ , which are isomorphic to the socle in our case. Second, we compare the middle-level subquotient  $\text{Mid}/\text{Soc}$  in the filtration (6.26) consisting of all subquotients/nodes having in-arrows from the top and out-arrows directed to the bottom/socle. The middle level of  $\mathcal{Y}_0$ , constituting of two copies for each  $F_{2j,-1}^{(0)}$  (see Fig. 7), has the same  $\mathcal{V}(2) \boxtimes \bar{\mathcal{V}}(2)$  content as the  $\mathcal{V}(2)$ -module  $\mathcal{H}$  has at the  $\mathfrak{h} = 0$  grade, where we use (6.27) and (2.12). Similarly one can proceed for any  $\mathfrak{h} = j$  grade.

The Kausch's  $\mathfrak{sl}(2)$  action discussed in Sec. 5 of the first paper [21], see also Sec. 6.3, ‘splits’ the sum (6.6) into sectors (direct summands) of different isospins<sup>14</sup> – e.g., with  $k \geq \frac{j}{2}$  and  $j \geq 1$ , in the

<sup>14</sup>In principle, it is possible to find the  $\mathfrak{sl}(2)$  content of the  $\mathcal{Y}_j$  modules in Fig. 7 using the lattice realization of the Kausch's  $\mathfrak{sl}(2)$  given in Sec. 5.3 in [21] but we do not do it in this paper.

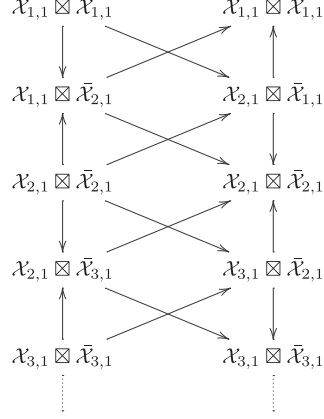


Figure 8: Two-strands structure of the vacuum sector for non-chiral symplectic fermions.

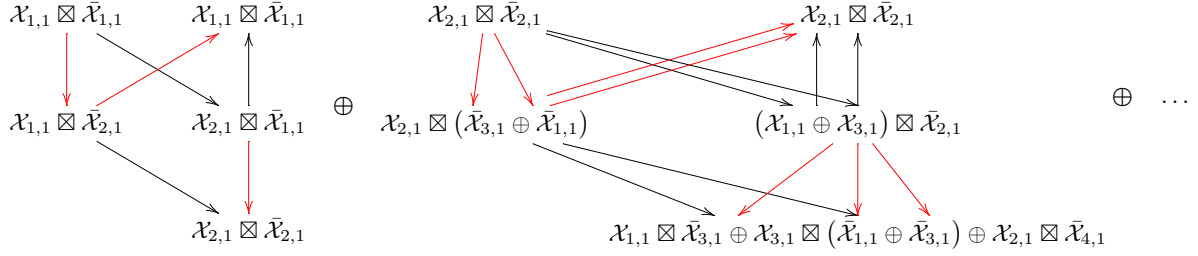
bosonic case. For each sector, the formulas (6.23) and (6.24) give the indecomposable structure under  $\mathcal{V}(2)$ . The simplest is for  $k = 0$ , where we obtain the structure represented on the left in Fig. 5. By a slight change of ‘geometry’ the diagram for this  $\mathcal{V}(2)$ -module can be represented as well as on Fig. 8, with a pattern of arrows coincident with the one in Fig. 7. The case  $k = 1$  in Fig. 6 appears more complicated, but it is only because of additional ‘gaps’ in the arrows when compared to  $k = 0$ . The decomposition is still fully compatible with Fig. 7. Here, additional checks of consistency between structures of  $\mathcal{Y}_j$  and  $\mathcal{H}_{(j)}$  involve finer filtrations. Consider for example the following filtration of  $U\mathfrak{S}_\infty$ -submodules in  $\mathcal{Y}_j$

$$0 = \mathcal{F}_{j-1} \subset \mathcal{F}_j \subset \mathcal{F}_{j+1} \subset \mathcal{F}_{j+2} \subset \cdots \subset \mathcal{Y}_j, \quad (6.28)$$

where  $\mathcal{F}_j$  is the submodule generated from the top subquotients  $F_{k,(-1)^{k+1}}^{(0)}$ , with  $k \leq j$ , in Fig. 7. It turns out that each term of this filtration is a direct sum of indecomposable  $\mathcal{V}(2)$ -modules, belonging in general to different isospin-sectors. This follows from the fact that the algebra  $U_{\mathfrak{q}}^{\text{odd}}\mathfrak{sl}(2)$  of all intertwining operators between  $U\mathfrak{S}_\infty$ -modules gives also intertwining operators for  $\mathcal{V}(2)$ -modules. To show an explicit example of a correspondence between the terms of the filtration (6.28) and  $\mathcal{V}(2)$ -modules, we consider for simplicity again the case  $j = 0$ , where the first non-trivial term in (6.28) is  $\mathcal{F}_1$  with the subquotient structure

$$\begin{array}{ccc}
 F_{1,1}^{(0)} & & F_{1,1}^{(0)} \\
 \downarrow & \nearrow & \uparrow \\
 F_{2,-1}^{(0)} & & F_{2,-1}^{(0)} \\
 & \searrow & \downarrow \\
 & & F_{3,1}^{(0)}
 \end{array} \quad (6.29)$$

It is decomposed over  $\mathbf{V}(2)$  into the direct sum (over all integer isospins)



where the first and second summands are particular  $\mathbf{V}(2)$ -submodules of  $\mathcal{H}_0^+$  in Fig. 8 and  $\mathcal{H}_1^+$  in Fig. 6, respectively (we just rearranged nodes introducing red arrows for the horizontal arrows in the diagrams in Fig. 6 and in Fig. 8 denoting the right Virasoro action). The other direct summands of higher isospins appear in the same pattern (6.29) with arrows and nodes in the  $U\mathfrak{S}_\infty$ -picture split according to (6.2) in order to get the left-right Virasoro diagrams. All the other terms  $\mathcal{F}_j$  in the filtration (6.28) can in principle be analyzed in a similar way but pictures are more complicated and we do not give them.

To conclude, we checked that all arrows present in the diagrams for the subquotient structure of the scaling limits of the spin-chain modules  $\hat{\mathcal{P}}_j$  or indecomposables  $\mathcal{Y}_j$  over  $U\mathfrak{S}_\infty$  are also in the corresponding infinite diagrams for the subquotient structure of modules over  $U\bar{\mathfrak{S}}_\infty$ , which contains Virasoro, and vice versa, confirming our earlier results. Our analysis was based on a decomposition onto direct summands over the left-right Virasoro. Then, indecomposable modules over the interchiral algebra  $\mathcal{S}$  are appropriate completions, where only each direct summand over  $\mathbf{V}(2)$  should be completed, such that it contains images of  $\bar{\mathfrak{gl}}_\infty$  generators and therefore vectors produced by the local operators  $S_n^{(l)}$  (recall that we have an embedding of  $\mathfrak{S}$  into  $\bar{\mathfrak{gl}}_\infty$ ). These completions can be also described with the use of the formal topology, as in Sec. 6.3.

## 7 Conclusion

This paper terminates our series on the scaling limit of the  $\mathfrak{gl}(1|1)$  spin chain. The conclusions of the analysis of [7] in the open case do carry over to the periodic case, but at the price of several complications. The bimodule structure of the spin chain over  $JTL_N$  and its centralizer  $U_i^{\text{odd}}\mathfrak{sl}(2)$  does turn out to be compatible with the known symplectic fermion continuum limit. However, simple modules over  $JTL_N$  correspond to direct sums of modules over  $\mathbf{V}(2)$ , the left-right Virasoro algebra, while the known  $\mathfrak{sl}(2)$  symmetry of Kausch is not present on the discrete spin chain in the bulk case (in contrast with the boundary case). The lesson drawn is that the good organizing object of the LCFT is truly the scaling limit of the  $JTL_N$  algebra, and that it contains more than  $\mathbf{V}(2)$ , which must be extended by non chiral fields (not commuting with  $\mathfrak{sl}(2)$ ), giving rise to the **interchiral algebra**  $\mathcal{S}$ . The centralizer of this interchiral algebra remains  $U_i^{\text{odd}}\mathfrak{sl}(2)$  in the scaling limit, and the bimodule structure of the  $\mathfrak{gl}(1|1)$  discrete spin chain carries over identically now to the scaling limit, exactly as in the open case. Interestingly, we note that the bimodule over the pair of algebras  $\mathcal{S}$  and  $U_i^{\text{odd}}\mathfrak{sl}(2)$  produces acyclic complexes with the differentials  $E_n$  and  $F_n$  ( $E_n^2 = F_n^2 = 0$ ). These can be interpreted as non-chiral analogues of the well-known Felder resolutions for chiral Virasoro representations.

The jaded reader might argue of course that the complex algebraic analysis presented here and in [21, 22] is not really necessary nor useful to understand most physical properties of symplectic fermions.

The point however is that we are in the process of developing a strategy to tackle interacting models for which very little is known, and a direct solution based on a (free) action simply unavailable. In this case, we believe that the algebraic analysis is an essential tool to make progress, and answer such simple questions as which conformal fields are degenerate and which are not, which fields have logarithmic partners, what are the logarithmic couplings, *etc.* It is also probably possible to obtain information about fusion rules using this approach.

Sadly, things are bound to be more complicated in most cases than they were for  $\mathfrak{gl}(1|1)$ . The reason is, that for  $\mathfrak{gl}(1|1)$ , the representation of  $JTL_N$  is in fact non-faithful – the structure of the bulk and boundary theories are, as a result, quite similar. General  $\mathfrak{gl}(n|m)$  or  $\mathfrak{osp}(n|2m)$  spin chains, such as those necessary to study the cases  $c = 0$ ,  $c = 1$ , will provide faithful representations, for which the structure of the indecomposable  $JTL_N$  modules will be considerably more involved. Their analysis will, in fact, require use of more sophisticated algebraic techniques, and will be started in [35]. Note that even for  $c = -2$ , the  $\mathfrak{gl}(1|1)$  spin chain is non generic, and the  $\mathfrak{gl}(2|2)$  spin chain for instance will lead, in the scaling limit, to a  $c = -2$  LCFT that bears little resemblance with symplectic fermions.

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## Appendix A: Fourier transforms

It is convenient to introduce Fourier transforms of the  $f_j$  and  $f_j^\dagger$  fermions used in the definition of our  $JTL_N$  representation in (2.2) and (3.1). We set, for  $1 \leq m \leq N$  (recall that we set  $N = 2L$ ),

$$\theta_{p_m} = \frac{1}{\sqrt{N}} \sum_{k=1}^N e^{-ikp_m} i^{-k} f_k, \quad \theta_{p_m}^\dagger = \frac{1}{\sqrt{N}} \sum_{k=1}^N e^{ikp_m} i^{-k} f_k^\dagger \quad (\text{A1})$$

with the set of allowed momenta

$$p_m = \begin{cases} \frac{2\pi m}{N}, & L - \text{even}, \\ \frac{(2m-1)\pi}{N}, & L - \text{odd}, \end{cases} \quad 1 \leq m \leq N, \quad (\text{A2})$$

and with the usual anti-commutation relations

$$\{\theta_{p_1}, \theta_{p_2}^\dagger\} = \delta_{p_1, p_2}, \quad \{\theta_{p_1}, \theta_{p_2}\} = \{\theta_{p_1}^\dagger, \theta_{p_2}^\dagger\} = 0.$$

We then introduce the following linear combinations of fermions

$$\begin{aligned}\chi_p^\dagger &= \frac{1}{\sqrt{2}} \left( \sqrt{\tan \frac{p}{2}} \theta_{p-\frac{\pi}{2}}^\dagger + \sqrt{\cot \frac{p}{2}} \theta_{p+\frac{\pi}{2}}^\dagger \right), & \chi_p &= \frac{1}{\sqrt{2}} \left( \sqrt{\cot \frac{p}{2}} \theta_{p-\frac{\pi}{2}} + \sqrt{\tan \frac{p}{2}} \theta_{p+\frac{\pi}{2}} \right), \\ \eta_p^\dagger &= \frac{1}{\sqrt{2}} \left( \sqrt{\tan \frac{p}{2}} \theta_{p-\frac{\pi}{2}}^\dagger - \sqrt{\cot \frac{p}{2}} \theta_{p+\frac{\pi}{2}}^\dagger \right), & \eta_p &= \frac{1}{\sqrt{2}} \left( \sqrt{\cot \frac{p}{2}} \theta_{p-\frac{\pi}{2}} - \sqrt{\tan \frac{p}{2}} \theta_{p+\frac{\pi}{2}} \right), \\ \chi_0^\dagger &= \theta_{\frac{\pi}{2}}^\dagger, \quad \chi_0 = \theta_{\frac{\pi}{2}}, & \eta_0^\dagger &= \theta_{\frac{3\pi}{2}}^\dagger, \quad \eta_0 = \theta_{\frac{3\pi}{2}},\end{aligned}\tag{A3}$$

with momenta  $p$  shifted by  $\pi/2$  and taking thus values  $p = p_n = \epsilon n$ , where  $1 \leq n \leq L-1$ , for even and odd  $L$ . The  $\chi$  and  $\eta$  fermions satisfy the anti-commutation relations

$$\{\chi_p^\dagger, \chi_{p'}\} = \{\eta_p^\dagger, \eta_{p'}\} = \delta_{p,p'}, \quad \{\chi_p, \eta_{p'}\} = \{\chi_p^\dagger, \eta_{p'}^{(\dagger)}\} = \{\eta_p^\dagger, \chi_{p'}^{(\dagger)}\} = 0.\tag{A4}$$

These  $\chi$  and  $\eta$  fermions are creation and annihilation operators for the Hamiltonian  $H$  from (3.2) and they were found in [21].

## Appendix B: Lie algebras $\mathfrak{gl}_\infty$ , $\overline{\mathfrak{gl}}_\infty$ , and $\mathfrak{S}$

Recall [42] that  $\mathfrak{gl}_\infty$  is a Lie algebra of infinite matrices with a *finite* number of non-zero elements. This algebra admits a well-known representation (so-called basic representation) in the free fermion models. Having fermion modes  $\psi_n^2$  with their conjugates  $\psi_{-n}^1$ , the basis elements  $\mathcal{E}_{n,m}$  of  $\mathfrak{gl}_\infty$  – the unit matrices having identity at the position  $(n, m)$  and zeros otherwise – can be written as

$$\mathcal{E}_{n,m} = \psi_n^2 \psi_{-m}^1, \quad n, m \in \mathbb{Z}.\tag{B1}$$

The commutation relations in this algebra are then the usual ones as in a finite  $\mathfrak{gl}_N$  but now indices are in  $\mathbb{Z}$ . The algebra  $\mathfrak{gl}_\infty$  admits a central extension  $\mathfrak{gl}_\infty \oplus \mathbb{C}\mathbf{1}$  which we denote by  $\mathfrak{gl}'_\infty$ . This extension is defined using the two-cocycle  $c : \mathfrak{gl}_\infty \times \mathfrak{gl}_\infty \rightarrow \mathbb{C}$  given by (see [42])

$$c(\mathcal{E}_{n,m}, \mathcal{E}_{m,n}) = -c(\mathcal{E}_{m,n}, \mathcal{E}_{n,m}) = 1, \quad \text{if } n \leq 0, m \geq 1, \quad \text{and} \quad c(\mathcal{E}_{n,m}, \mathcal{E}_{k,l}) = 0 \quad \text{otherwise}.\tag{B2}$$

Note that  $c$  satisfies the cocycle condition that allows to modify the commutation relations in  $\mathfrak{gl}_\infty$  as

$$[a + x\mathbf{1}, b + y\mathbf{1}] = [a, b] + c(a, b)\mathbf{1}, \quad a, b \in \mathfrak{gl}_\infty, \quad x, y \in \mathbb{C}.\tag{B3}$$

The completed algebra  $\overline{\mathfrak{gl}}_\infty$  is then defined as a Lie algebra of infinite matrices with a possibly infinite number of non-zero elements but the matrix still has a finite number of non-zero diagonals, those which are along the main diagonal containing the Cartan elements  $\mathcal{E}_{n,n}$ . This completion can be also described using a formal algebraic completion of the Cartan subalgebra in  $\mathfrak{gl}_\infty$ , see [42].

Recall that we introduced the Lie algebra  $\mathfrak{S}$  in the end of Sec. 5.4.3 – it is a Lie algebra of our local operators. We then define an embedding of  $\mathfrak{S}$  (and also  $\overline{\mathfrak{S}}$ ) into  $\overline{\mathfrak{gl}}_\infty$  in the following way. We first introduce new fermionic modes  $\psi_m^\alpha$ , for  $\alpha = 1, 2$  and  $m \in \mathbb{Z}$ , and reorganize the modes of the left and right symplectic fermions,  $\psi_n^\alpha$  and  $\bar{\psi}_n^\alpha$ , such that the left modes are identified with even modes of  $\psi$ 's and the right fermionic modes correspond to odd modes of  $\psi$ 's:

$$\psi_n^\alpha = \psi_{2n}^\alpha, \quad \bar{\psi}_n^\alpha = \psi_{-2n+1}^\alpha, \quad n \in \mathbb{Z}, \quad \alpha = 1, 2.\tag{B4}$$

Recall that we also have a pair of modes  $\phi_0^{1,2}$  conjugate to  $\psi_0^{2,1}$  – we do not use new notations for them. Under the identification (B4), it is then easy to see that the scaling limit of our  $\mathfrak{gl}_N$  algebras described in Sec. 5.1.2, and in more precise terms in App. C, is given by the basic representation (B1) of  $\mathfrak{gl}_\infty$  where one should include also bilinears like  $\psi_n^2 \phi_0^1$ , *etc.* The Lie algebra  $\mathfrak{S}$  which has basis elements as infinite sums of  $\mathcal{E}_{n,m}$ 's can be now embedded into  $\overline{\mathfrak{gl}}_\infty$ . Indeed, the Virasoro-like generators  $L_n^{(l)}$  are the sums of bilinears in the  $\psi$  modes that have infinite number of terms like  $\psi_{2k}^2 \psi_{-2k+n}^1$  and all these terms belong to the same diagonal along the main one containing the Cartan elements  $\psi_k^2 \psi_{-k}^1$ . Therefore any finite linear combination of  $L_n^{(l)}$  belongs to  $\overline{\mathfrak{gl}}_\infty$ , and similarly for  $\bar{L}_n^{(l)}$ . Note then that the basis elements  $S_n^{(l)}$  are given by the sums containing infinite number of terms like  $\psi_{2k}^2 \psi_{-2k+1+n}^1$  and they also belong to  $\overline{\mathfrak{gl}}_\infty$ . Finally, we obtain that any linear and finite combination of basis elements in  $\mathfrak{S}$  belongs to  $\overline{\mathfrak{gl}}_\infty$ . We use this embedding  $\mathfrak{S} \hookrightarrow \overline{\mathfrak{gl}}_\infty$  in the main text for defining appropriate completions of our modules such that the completions contain images under the action of the local operators from the interchiral algebra  $\mathcal{S}$ .

## Appendix C: Direct and inverse limits

In order to satisfy the more formal reader, we give here precise definition of the Lie algebra  $\mathfrak{S}_\infty$  using direct-limit constructions. We first remind a formal definition of the direct limit. A direct (or inductive) system is a pair  $\{A_i, \phi_{ij}\}$  of a family of algebraic objects  $A_i$  (vector spaces, algebras, *etc.*) indexed by an ordered set  $I$  and a family of homomorphisms  $\phi_{ij} : A_i \rightarrow A_j$  for all  $i \leq j$  satisfying the following properties: (i)  $\phi_{ii}$  is the identity on  $A_i$ , and (ii)  $\phi_{ik} = \phi_{jk} \phi_{ij}$  for all  $i \leq j \leq k$ . The *direct limit*  $A_\infty \equiv \varinjlim A_i$  of the direct system  $\{A_i, \phi_{ij}\}$  is defined as the disjoint union  $\coprod_i A_i / \sim$  modulo an equivalence relation: two elements  $a_i \in A_i$  and  $a_j \in A_j$  in the disjoint union are equivalent if and only if they eventually become equal in the direct system, *i.e.*,  $a_i \sim a_j$  if there is  $k \in I$  such that  $\phi_{ik}(a_i) = \phi_{jk}(a_j)$ . We obtain from this definition canonical homomorphisms  $\phi_i : A_i \rightarrow A_\infty$  mapping each element to its equivalence class. The algebraic operations on  $A_\infty$  are then defined using these maps. Recall also that an inverse or projective system of algebraic objects and its inverse limit can be defined in a similar way reverting order and arrows everywhere.

Consider then a case of even  $L = N/2$  for simplicity and an embedding  $\phi_L$  of the Clifford algebra  $\mathcal{C}_{4L}$  into a bigger one  $\mathcal{C}_{4(L+2)}$  defined by

$$\phi_L(\chi_p^{(\dagger)}) = \chi_{p'}^{(\dagger)}, \quad \phi_L(\eta_p^{(\dagger)}) = \eta_{p'}^{(\dagger)}, \quad \text{for } 0 \leq m \leq L/2 - 1, \quad (\text{C1})$$

$$\phi_L(\chi_{\pi-p}^{(\dagger)}) = \chi_{\pi-p'}^{(\dagger)}, \quad \phi_L(\eta_{\pi-p}^{(\dagger)}) = \eta_{\pi-p'}^{(\dagger)}, \quad \text{for } 1 \leq m \leq L/2, \quad (\text{C2})$$

where we set  $p \equiv p(m) = \frac{m\pi}{L}$ , with  $0 \leq m \leq L-1$ , and for the indexing set in the bigger Clifford algebra we set  $p' \equiv p'(m) = \frac{m\pi}{L+2}$ , with  $0 \leq m \leq L+1$ . The embedding  $\phi_L$  induces the corresponding embedding of the module  $\mathcal{H}_{2L}$  over  $\mathcal{C}_{4L}$  into the module  $\mathcal{H}_{2(L+2)}$ , *e.g.*,  $\phi_L(\chi_p^\dagger \Omega) = \chi_{p'}^\dagger \Omega$  only for  $0 \leq m \leq L/2 - 1$ , *etc.*, where we denote the embeddings of the representation spaces of the Clifford algebras by the same letter  $\phi_L : \mathcal{H}_{2L} \rightarrow \mathcal{H}_{2(L+2)}$ . We thus construct two direct systems: the one for the Clifford algebras

$$\mathcal{C}_4 \xrightarrow{\phi_1} \mathcal{C}_{12} \xrightarrow{\phi_3} \dots \xrightarrow{\phi_{L-2}} \mathcal{C}_{4L} \xrightarrow{\phi_L} \mathcal{C}_{4(L+2)} \xrightarrow{\phi_{L+2}} \dots \quad (\text{C3})$$

with the direct limit an infinite-dimensional Clifford algebra  $\mathcal{C} \equiv \varinjlim_L \mathcal{C}_{4L}$  generated by the symplectic fermions modes  $\psi_n^{1,2}$ ,  $\bar{\psi}_m^{1,2}$ , and  $\phi_0^{1,2}$  introduced in (5.5) and (5.6); the second direct system is for the

spin-chains

$$\mathcal{H}_2 \xrightarrow{\phi_1} \mathcal{H}_6 \xrightarrow{\phi_3} \dots \xrightarrow{\phi_{L-2}} \mathcal{H}_{2L} \xrightarrow{\phi_L} \mathcal{H}_{2(L+2)} \xrightarrow{\phi_{L+2}} \dots \quad (\text{C4})$$

The direct limit  $\varinjlim_L \mathcal{H}_{2L}$  then coincides with the space  $\mathcal{H}$  of scaling states or finite-energy and finite-spin states which was described in Sec. 5.1.1.

Notice then that the direct system of Clifford algebras defines a direct system of the matrix algebras  $\mathfrak{gl}_{2L}$  in the following way<sup>15</sup>

$$\dots \xrightarrow{\phi_{L-2}} \begin{pmatrix} \mathcal{E}_{m,n} & \mathcal{E}_{m,L+n} \\ \mathcal{E}_{L+m,n} & \mathcal{E}_{L+m,L+n} \end{pmatrix}_{2L \times 2L} \xrightarrow{\phi_L} \begin{pmatrix} 0 & 0 \dots 0 & 0 & 0 & 0 \dots 0 & 0 \\ 0 & \mathcal{E}_{m,n} & 0 & 0 & \mathcal{E}_{m,L+n} & 0 \\ 0 & 0 \dots 0 & 0 & 0 & 0 \dots 0 & 0 \\ 0 & 0 \dots 0 & 0 & 0 & 0 \dots 0 & 0 \\ 0 & \mathcal{E}_{L+m,n} & 0 & 0 & \mathcal{E}_{L+m,L+n} & 0 \\ 0 & 0 \dots 0 & 0 & 0 & 0 \dots 0 & 0 \end{pmatrix}_{2(L+2) \times 2(L+2)} \xrightarrow{\phi_{L+2}} \dots$$

where the elements  $\mathcal{E}_{i,j}$  denote usual unit matrices, with  $0 \leq n, m \leq L-1$ , defined in (4.5) and we abuse notations denoting the embeddings of matrix algebras by the same  $\phi_L$  as for the Clifford algebras. The direct limit  $\varinjlim_L \mathfrak{gl}_{2L}$  then gives the infinite-dimensional Lie algebra that we call  $\mathfrak{gl}_\infty$  – an algebra of infinite matrices with a finite number of non-zero elements.

Using Thm. 4.4 and Dfn. 4.3.1, we could define the scaling limit of JTL algebras using the direct system for  $\mathfrak{gl}_{2L}$  above which gives a direct system of embeddings for the Lie subalgebras  $\mathfrak{sp}'_{2L-2}$  and  $\mathfrak{S}'_N$ . Unfortunately, going this way we would get in the direct limit the infinite-dimensional Lie algebras without the central term. So, in order to get  $\mathfrak{sp}'_\infty$  and the  $\mathfrak{S}_\infty$  algebras containing the central term and having thus the proper basis, necessary for physics, we should slightly modify the embeddings  $\phi$ 's putting primes everywhere, *i.e.*, mapping the normally ordered generators of  $\mathfrak{sp}'_{2L-2}$  as  $\phi'_L(\mathcal{A}'_{m,n}) = \mathcal{A}'_{m,n}$  and setting  $\phi'_L(\mathbf{1}_N) = \mathbf{1}_{N+4}$ , where  $\mathbf{1}_N$  is the identity or the central element in  $\mathfrak{S}'_N = \mathfrak{S}_N \oplus \mathbb{C}\mathbf{1}_N$ . These  $\phi'_L$  give embeddings of the Lie algebras  $\mathfrak{S}'_N$  which differ from  $\phi_L$ . Now, this new direct system gives in the limit  $\varinjlim_L \mathfrak{S}'_{2L}$  what we call the  $\mathfrak{S}_\infty$  Lie algebra and the identity is mapped under the canonical homomorphisms  $\mathfrak{S}'_N \rightarrow \mathfrak{S}_\infty$  to the central element in  $\mathfrak{S}_\infty$ . Finally, on the level of enveloping algebras we get the direct system for (representations of) JTL algebras

$$\pi_{\mathfrak{gl}}(JTL_2) \xrightarrow{\phi'_1} \pi_{\mathfrak{gl}}(JTL_6) \xrightarrow{\phi'_3} \dots \xrightarrow{\phi'_{L-2}} \pi_{\mathfrak{gl}}(JTL_{2L}) \xrightarrow{\phi'_L} \pi_{\mathfrak{gl}}(JTL_{2(L+2)}) \xrightarrow{\phi'_{L+2}} \dots \quad (\text{C5})$$

where  $\phi'_i$  here are embeddings of the finite-dimensional associative algebras. Note that an existence of embeddings for JTL algebras is a non-trivial thing and it is very hard to write it in terms of diagrams. Nevertheless, we found a special basis in which a system of embeddings is easily constructed and the direct system (C5) gives the limit – an enveloping algebra of  $\mathfrak{S}_\infty$  – which we call *the scaling limit of JTL algebras*.

It turns out that the Lie algebra  $\mathfrak{S}_\infty$  has  $T_{m,n}$ ,  $\bar{T}_{m,n}$  and  $S_{m,n}$  with  $m, n \in \mathbb{Z}$  as its basis or equivalently the universal enveloping algebra  $U\mathfrak{S}_\infty$  has the symplectic-fermion representation on  $\mathcal{H}$  generated by  $\mathbf{1}$  and

$$S_{n,m} = S_{\alpha\beta} \psi_n^\alpha \bar{\psi}_m^\beta, \quad T_{k,l} = J_{\alpha\beta} \psi_k^\alpha \psi_l^\beta, \quad \bar{T}_{r,s} = J_{\alpha\beta} \bar{\psi}_r^\alpha \bar{\psi}_s^\beta, \quad \text{with } n, m, k, l, r, s \in \mathbb{Z}, \quad (\text{C6})$$

<sup>15</sup>After actually rearranging four rows and four columns of zeros for a simpler presentation which is just a matter of conventions.

as was also pointed out in Sec. 5.1.2.

Further, on the side of centralizers, we construct inverse systems for the spin-chain representations of  $\mathfrak{J}_{JTL}$  in the usual way [45, 10] as it is done for the  $\mathfrak{q}$ -Schur algebras for  $U_{\mathfrak{q}}sl(2)$ . Then, the inverse limit defines an infinite-dimensional associative algebra which we identify with a quotient of  $U_{\mathfrak{q}}^{\text{odd}}sl(2)$  that admits among irreducible representations only one-dimensional ones, see representation theory of  $U_{\mathfrak{q}}^{\text{odd}}sl(2)$  in [22]. We call this inverse limit as *the scaling limit of the JTL centralizers*  $\mathfrak{J}_{JTL}$ . We will denote this quotient also by  $U_{\mathfrak{q}}^{\text{odd}}sl(2)$  which should not confuse the reader because we will specify the space it acts on. The fermionic expressions for the generators in the scaling limit of the centralizers were computed in our first paper [21] and they are

$$E_n = \left[ \sum_{m>0} \left( \frac{\psi_m^2 \psi_{-m}^2}{m} - \frac{\bar{\psi}_m^2 \bar{\psi}_{-m}^2}{m} \right) \right]^n \psi_0^2, \quad F_n = \left[ \sum_{m>0} \left( \frac{\psi_m^1 \psi_{-m}^1}{m} - \frac{\bar{\psi}_m^1 \bar{\psi}_{-m}^1}{m} \right) \right]^n \psi_0^1,$$

with the representation of the Cartan element  $\mathfrak{h}$  as

$$\mathfrak{h} = -i/2(\psi_0^2 \phi_0^1 + \psi_0^1 \phi_0^2) + \sum_{m>0} \frac{1}{m} (\psi_{-m}^2 \psi_m^1 + \psi_{-m}^1 \psi_m^2 + \bar{\psi}_{-m}^2 \bar{\psi}_m^1 + \bar{\psi}_{-m}^1 \bar{\psi}_m^2)$$

while the generator  $K = (-1)^{2\mathfrak{h}}$ . Note finally that the central term of  $\mathfrak{S}_{\infty}$  belongs also to the center of  $U_{\mathfrak{q}}^{\text{odd}}sl(2)$ .

## Appendix D: Emergence of the interchiral algebra from the lattice

We look at the Fourier transforms of  $e_j$ 's introducing  $q = n\pi/L$ , for  $n \in \mathbb{Z}$ , and defining the operators

$$H(n) = - \sum_{j=1}^N e^{-iqj} e_j = \sum_p \left[ 1 + e^{iq} + ie^{-ip} - ie^{i(p+q)} \right] \theta_p^{\dagger} \theta_{p+q+\pi} \quad (\text{D1})$$

which were obtained in our first paper [21]. These can be rewritten using the more convenient  $\chi$  and  $\eta$  fermions from (A3) as

$$H(n) = 2e^{iq/2} \left( \sqrt{\sin q} \chi_0^{\dagger} (\chi_q + \eta_q) + \sum_{\substack{p=\epsilon \\ \text{step}=\epsilon}}^{\pi-q-\epsilon} \sqrt{\sin(p) \sin(p+q)} (\chi_p^{\dagger} \chi_{p+q} - \eta_p^{\dagger} \eta_{p+q}) \right. \\ \left. + \sum_{\substack{p=\epsilon \\ \text{step}=\epsilon}}^{q-\epsilon} \sqrt{\sin(p) \sin(q-p)} (\chi_{\pi-p}^{\dagger} \eta_{q-p} + \eta_{\pi-p}^{\dagger} \chi_{q-p}) + \sqrt{\sin q} (\chi_{\pi-q}^{\dagger} + \eta_{\pi-q}^{\dagger}) \eta_0 \right), \quad (\text{D2})$$

where  $0 < n < L$ , while a similar expression can be written for other values of the modes  $n$ .

In order to take the scaling limit for  $n$  close to  $L$  we define  $k = L - n$  and consider operators  $\tilde{H}(k) = H(L - k)$  and study their scaling limit for finite  $k$ . We thus first substitute  $n \rightarrow L - k$  and  $q \rightarrow \pi - q'$ , with  $q' = k\pi/L$ , in (D2). Then, using the limit (5.7)-(5.8) to the fermions  $\psi^{1,2}$  and  $\bar{\psi}^{1,2}$  and linearizing the dispersion relation, we obtain in the scaling limit (keeping terms in the sums with



the momentum  $p$  close to 0 or to  $\pi$ ), with finite positive  $k$ ,

$$\begin{aligned} -i\frac{L}{2\pi}\tilde{H}(k) \mapsto & \psi_0^2(\bar{\psi}_k^1 - \psi_{-k}^1) + \sum_{m=1}^{k-1} (\psi_{-m}^2 \bar{\psi}_{k-m}^1 - \bar{\psi}_m^2 \psi_{m-k}^1) + \sum_{m \geq k+1} (\psi_{-m}^2 \bar{\psi}_{k-m}^1 - \bar{\psi}_m^2 \psi_{m-k}^1) \\ & + \sum_{m \geq 1} (\psi_m^2 \bar{\psi}_{m+k}^1 - \bar{\psi}_{-m}^2 \psi_{-m-k}^1) + (\psi_{-k}^2 - \bar{\psi}_k^2) \psi_0^1. \end{aligned}$$

Gathering all terms and using the relations (5.5), we finally obtain the contribution corresponding to low-lying excitations over the ground state,

$$\frac{L}{2\pi i}\tilde{H}(k) \mapsto S_{\alpha\beta} \sum_{m \in \mathbb{Z}} \psi_m^\alpha \bar{\psi}_{m+k}^\beta \equiv S_k, \quad k > 0,$$

which we denote in what follows by  $S_k$  or  $S_k^{(0)}$ . A similar computation gives the same scaling limit of  $\frac{L}{2\pi i}\tilde{H}(k)$  for negative finite  $k$ .

We then consider the Fourier transformation of the momentum operator  $P$ ,

$$P(n) = \frac{i}{2} \sum_{j=1}^N e^{-iqj} [e_j, e_{j+1}], \quad q = \frac{n\pi}{L},$$

which gives [21] on a finite spin-chain the expression in  $\chi$ - $\eta$  fermions:

$$\begin{aligned} P(n) = & 2e^{iq} \left( \cos \frac{q}{2} \sqrt{\sin q} \chi_0^\dagger (\chi_q - \eta_q) + \sum_{\substack{p=\epsilon \\ \text{step}=\epsilon}}^{\pi-q-\epsilon} \cos \left(p + \frac{q}{2}\right) \sqrt{\sin(p) \sin(p+q)} (\chi_p^\dagger \chi_{p+q} + \eta_p^\dagger \eta_{p+q}) \right. \\ & \left. - \sum_{\substack{p=\epsilon \\ \text{step}=\epsilon}}^{q-\epsilon} \cos \left(p - \frac{q}{2}\right) \sqrt{\sin(p) \sin(q-p)} (\chi_{\pi-p}^\dagger \eta_{q-p} - \eta_{\pi-p}^\dagger \chi_{q-p}) - \cos \frac{q}{2} \sqrt{\sin q} (\chi_{\pi-q}^\dagger - \eta_{\pi-q}^\dagger) \eta_0 \right) \end{aligned}$$

Once again, to take the scaling limit for  $n$  close to  $L$  we consider operators  $\tilde{P}(k) = P(L-k)$  and study their scaling limit for finite  $k$ . It turns out that the scaling limit (for finite  $k$ ) is

$$-\frac{L^2}{\pi^2} \tilde{P}(k) \mapsto S_{\alpha\beta} \sum_{m \in \mathbb{Z}} (2m+k) \psi_m^\alpha \bar{\psi}_{m+k}^\beta \equiv S_k^{(1)}, \quad k \in \mathbb{Z}, \quad (\text{D3})$$

where the operators  $S_k^{(l)}$  are introduced in (5.39), for  $l \geq 0$  and  $k \in \mathbb{Z}$ .

## D.1 Higher Hamiltonians and the Lie algebra $\mathfrak{S}_N$

Recall that in Sec. 4 we gave a Lie algebraic  $\mathfrak{S}_N$  description of the JTL algebra in terms of fermion bilinears. We consider here a different basis in  $\mathfrak{S}_N$  that is spanned by the generalized higher Hamiltonians  $H_l(n)$  introduced (5.31). These Hamiltonians for positive modes  $n$  were expressed in (5.33) in the basis of  $\mathfrak{S}_N$ . In order to check that the linear span of all  $H_l(n)$ 's gives the vector space  $\mathfrak{S}_N$  we

need also negative-mode expressions. For  $l \geq 0$  and  $1 \leq n \leq L$ , we find a slightly different formula

$$H_l(-n) = 2e^{-iq\frac{l+1}{2}} \left( \sum_{m=n+1}^{L-1} \cos l(p - \frac{q}{2}) \sqrt{\sin(p) \sin(p-q)} \mathcal{A}_{m,m-n} \right. \\ \left. - \frac{1}{2} \sum_{m=1}^{n-1} \cos l(p - \frac{q}{2}) \sqrt{\sin(p) \sin(q-p)} (\mathcal{C}_{L-m,L+m-n} + (-1)^l \mathcal{B}_{m,n-m}) \right. \\ \left. + \cos \frac{lq}{2} \sqrt{\sin q} (\mathcal{E}_{0,L-n} - (-1)^l \mathcal{E}_{0,L+n} + \mathcal{E}_{n,L} - (-1)^l \mathcal{E}_{2L-n,L}) \right). \quad (\text{D4})$$

We checked then up to  $N = 20$  that a basis in the linear span of all  $H_l(n)$ 's, with  $l \geq 0$  and  $n \in \mathbb{Z}$ , resides actually in the range  $-\pi + l + 1 \leq q = n\pi/L \leq \pi + l + 2$  and  $l$  – the index for the family – runs from 0 to  $L - 1$ . Moreover, the number of linearly independent  $H_l(n)$ 's in this range is given by the dimension of the Lie algebra  $\mathfrak{S}_N$  introduced in Dfn. 4.3.1. We thus conclude that the  $H_l(n)$ 's should be closed under commutations (which is really hard to compute explicitly) and that they just give a different basis in the Lie algebra  $\mathfrak{S}_N$ .

## Appendix E: The character formulas

We provide here an analysis of the left-right Virasoro  $\mathcal{V}(2) = \mathcal{V}(2) \boxtimes \bar{\mathcal{V}}(2)$  content of the simple  $JTL_N$  modules in the scaling limit using a general strategy (via the XXZ spin chain) that will be useful for other models as well. We recall the basic fact [46] that

$$\text{Tr} e^{-\beta_R(H - Ne_0)} e^{-i\beta_I P} \rightarrow \text{Tr} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \quad (\text{E1})$$

where  $H$  and  $P$  are a lattice Hamiltonian (normalized such that the velocity of sound is unity) and momentum (see [21, 22] and below),  $e_0$  is the ground state energy per site in the scaling limit,  $q(\bar{q}) = \exp[-\frac{2\pi}{N}(\beta_R \pm i\beta_I)]$  with  $\beta_{R,I}$  real and  $\beta_R > 0$ , and  $N$  is the length of the chain. On the right-hand side, we have  $L_0$  and  $\bar{L}_0$  as zero modes of the stress-energy tensor, and  $c$  is the corresponding central charge. The trace on the left is taken over a subspace (of scaling states) of the spin chain, and the trace on the right is over the states occurring in this subspace in the scaling limit.

The calculation of an expression such as the trace on the left of (E1) is most easily done when the spin chain is the well known XXZ chain with appropriate value of the deformation parameter  $\mathfrak{q}$  and appropriately twisted boundary conditions. To be more specific, we consider the Hamiltonian

$$H_{\text{XXZ}}(K) = \sum_{j=1}^{2L-1} (\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+) + e^{2iK} \sigma_{2L}^+ \sigma_1^- + e^{-2iK} \sigma_{2L}^- \sigma_1^+ + \sum_{j=1}^{2L} \frac{\mathfrak{q} + \mathfrak{q}^{-1}}{4} \sigma_j^z \sigma_{j+1}^z, \quad (\text{E2})$$

where the  $2 \times 2$ -matrices  $\sigma^\pm$ ,  $\sigma^z$  are Pauli matrices. The momentum operator  $P_{\text{XXZ}}(K)$  can be chosen either by using the exact eigenvalues of the translation operator on the spin chain, or by using the general formulas in [21]. Parametrizing  $\mathfrak{q} = e^{i\pi/(x+1)}$ , it has been known for a long time thanks to Coulomb gas and Bethe ansatz arguments [8, 47] that

$$\text{Tr}_{\mathcal{H}_{[j]}} e^{-\beta_R(H_{\text{XXZ}}(K) - 2Le_0)} e^{-i\beta_I P_{\text{XXZ}}(K)} \rightarrow F_{j,e^{2iK}} \quad (\text{E3})$$

where

$$F_{j,e^{2iK}} = \frac{1}{q^{c/24}P(q)\bar{q}^{c/24}P(\bar{q})} \sum_{n \in \mathbb{Z}} q^{\frac{[xj+(x+1)(n+K)]^2-1}{4x(x+1)}} \bar{q}^{\frac{[xj-(x+1)(n+K)]^2-1}{4x(x+1)}}, \quad (\text{E4})$$

and the subspace  $\mathcal{H}_{[j]}$  is the subspace of spin  $S^z = j$ , with  $-L \leq j \leq L$ , of the spin-chain of length  $N = 2L$  (also denoted by  $\widehat{\mathcal{P}}_j$ ).

On the other hand, it is also well known [8] that this XXZ Hamiltonian can be written as

$$H_{\text{XXZ}}(K) = - \sum_{i=1}^{2L} e_i \quad (\text{E5})$$

where the  $e_i$  are expressed in the spin-1/2 basis [8] and, together with  $u^2$ , provide a representation of the Jones–Temperley–Lieb algebra  $JTL_N$  whenever  $e^{2ijK} = 1$ , for  $j \neq 0$ , and  $e^{2iK} = \mathfrak{q}^2$ , for  $j = 0$ . This representation has for dimension the usual binomial coefficient  $\binom{2L}{L+j}$ , and is believed to be isomorphic to the standard module  $\mathcal{W}_{[j],e^{\pm i\phi}}$  for  $\mathfrak{q}$  generic and  $\phi = -2K$ , where the ‘+’ is for a positive  $j$ , and the ‘−’ is for a negative value of  $j$ . The standard modules (defined and discussed in [22]) on opposite sectors  $S^z = \pm j$  are conjugate to each other with respect to the bilinear pairing  $\mathcal{W}_{[j],z} \times \mathcal{W}_{[j],z^{-1}} \rightarrow \mathbb{C}$  from [31] invariant with respect to the action of the affine TL algebra denoted there by  $T_N^a$ . The pairing has a trivial radical only in the generic cases.

When  $\mathfrak{q}$  is a root of unity, the representation obtained from the twisted XXZ chain is not isomorphic to a standard module any longer. But it is well established [8] that the traces over the modules and their scaling limit behave smoothly across those points. This means that we can use (E4) to obtain the generating function of conformal weights (that is, eigenvalues of  $L_0$  and  $\bar{L}_0$ ) in the scaling limit of the modules occurring in our  $\mathfrak{gl}(1|1)$  spin chain.

Let us specialize now to  $\mathfrak{q} = i$  ( $x = 1$ ) where  $c = -2$  and

$$F_{j,e^{2iK}} = \frac{1}{q^{-1/12}P(q)\bar{q}^{-1/12}P(\bar{q})} \sum_{n \in \mathbb{Z}} q^{\frac{[j+2(n+K)]^2-1}{8}} \bar{q}^{\frac{[j-2(n+K)]^2-1}{8}} \quad (\text{E6})$$

(note that  $F_{0,e^{2iK}} = F_{0,e^{-2iK}}$ ). Recall the Kac formula for the value of the central charge  $c = -2$

$$h_{r,s} = \frac{(2r-s)^2-1}{8}, \quad (\text{E7})$$

Introduce now the character of the Kac representation

$$K_{r,s} = \frac{q^{h_{rs}} - q^{h_{r,-s}}}{q^{c/24}P(q)} = \frac{q^{(2r-s)^2/8} - q^{(2r+s)^2/8}}{\eta(q)} \quad (\text{E8})$$

where the Dedekind’s eta function is  $\eta(q) = q^{1/24}P(q)$ . A short calculation establishes the crucial formula

$$F_{j,q^{2j+2k}} - F_{j+k,q^{2j}} = \sum_{r=1}^{\infty} K_{r,k} \bar{K}_{r,k+2j} \quad (\text{E9})$$

with  $\mathfrak{q} = i$  here.

From the structure of the spin chain modules (see Sec. 3.3 and Fig. 4), we deduce the traces

$$F_{j,(-1)^{j+1}}^{(0)} \equiv \lim_{L \rightarrow \infty} \text{Tr}_{\mathcal{L}_{j,(-1)^{j+1}}} e^{-\beta_R(H(0)-Le_0)} e^{-i\beta_I P(0)} \quad (\text{E10})$$

in each of the simple subquotients  $\mathcal{L}_{j,(-1)^{j+1}}$  appearing in the (scaling limit of)  $JTL$ -modules  $\widehat{\mathcal{P}}_k$ :

$$F_{j,(-1)^{j+1}}^{(0)} = \sum_{j' \geq j} (-1)^{j'-j} \sum_{r=1}^{\infty} K_{r,1} \bar{K}_{r,2j'+1}, \quad j \leq k. \quad (\text{E11})$$

On the other hand, recall the characters of the simples of Virasoro at  $c = -2$

$$\chi_{j,1} = \frac{q^{(2j-1)^2/8} - q^{(2j+1)^2/8}}{\eta(q)} \quad (\text{E12})$$

from which it follows that

$$K_{r,2j+1} = \sum_{s=-r+1}^r \chi_{j+s,1}. \quad (\text{E13})$$

Moreover, we formally define the character when  $r$  is negative by the same formula, so we have  $\chi_{-j,1} = -\chi_{j,1}$ . It follows that

$$K_{r,1} = \chi_{r,1}. \quad (\text{E14})$$

Straightforward algebra then leads to the key result

$$F_{j,(-1)^{j+1}}^{(0)} = \sum_{j_1, j_2 > 0}^* \chi_{j_1,1} \bar{\chi}_{j_2,1} \quad (\text{E15})$$

where the sum is done with the following constraints:

$$|j_1 - j_2| + 1 \leq j, \quad j_1 + j_2 - 1 \geq j, \quad j_1 + j_2 - j = 1 \pmod{2} \quad (\text{E16})$$

(note this is equivalent to treating  $j$  not as a spin but as a degeneracy, *i.e.*, setting  $j = 2s + 1$ ,  $j_i = 2s_i + 1$  and combining  $s_1$  and  $s_2$  to obtain spin  $s$ ).

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